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Electoral systems and the distortion of voters' preferences

Abstract

In this paper I show that in a parliamentary democracy, contrary to common wisdom, under a proportional electoral rule governments do not necessarily represent voters' preferences better than under plurality rule. While voters affect the composition of Parliament, decisions are taken by a subset of Parliamentarians: a coalition of them decides directly and through the government. As a consequence, two distortions might occur: one at the electoral stage when Parliament is formed and the other at the coalition formation stage, when government is chosen. Through a model la Rubinstein, I show that small parties' bargaining power increases when parties are patient; for sufficiently patient parties, the small (but pivotal) ones obtain a large bargaining power. The distortion introduced by plurality rule goes in the opposite direction; this can be beneficial (in term of voters' representativeness) as long as the impact of the two distortions is similar. I show that under non restrictive conditions, plurality rule can outperform the proportional rule in terms of representativeness of voters' preferences.

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"Thus there is an inherent conflict between two goals. The ideals of democracy and equality require as proportional representation as possible while efficient government often requires less proportional representation" (Laakso and Taagepera (1981), p. 107).

1 Introduction

Electoral systems differ from one other in several aspects. Among them, social scientists might focus on two that are particularly relevant: (1) government efficiency and (2) representativeness.

By efficiency, I mean the capability to produce well structured and coherent laws, wasting as few resources as possible. For that, the government needs to be stable over time and to take fast decisions. Efficiency is beyond the scope of this work. By representativeness, I mean the measure of the distance between legislator's and citizens' objective function. From that perspective, the best electoral system would attach to a party a power equal its share of voters. Here, I focus on representativeness, which means that I aim to minimise the difference between party's power and share of votes.

Under proportional electoral systems chances are good even for small parties to be represented in Parliament and to play an active role in the Government. This induces government instability,[1] since decisions must be accepted by a large number of parties, each having a (possibly different) agenda.[2]

Electoral systems are certainly the result of a country's culture. and it might be hard to define the causal relation between the number of parties in a country and the type of electoral system. Empirical evidence[3] nevertheless shows that proportional electoral systems tend to produce more fragmented political scenarios, with many parties represented in Parliament and large governmental coalitions. This increases the average time needed to produce laws and decreases the average duration of governments.

On the other hand, common wisdom and supporters of proportional systems[4]

[1] See Laakso and Taagepera (1981).

[2] On top of that, the proportional rule does not incentivise ideologically-close parties to cooperate. Proportional systems might even incentivise pre-electoral separation of existing parties into smaller parties, each close to a given group of voters, and to eventually find a post-election agreement. That way, they might be able to jointly receive more votes than if they were to run together. Studying that is beyond the scope of this paper.

[3] See Laakso and Taagepera (1981), Nurmi (1981) and Schofield (1981).

[4] See, for instance, Douglas (1923).

argue that voters' preferences are better represented and thus proportional systems are more equitable. The aim of this paper is to show that the claim that voters' preferences are better represented by proportional system is not necessarily true.

In Italy, for instance, from 2006 to 2008 Mr. Prodi led a centre-left government. Senators from a small pivotal party within the coalition heavily influenced his activity. This fact highlights the complexity of the link between voters' preferences and legislative outcome.

In representative democracies we observe the presence of two filters between principal (voters) and agent (legislator or government): (1) the electoral system and (2) the coalition formation process (see figure 1).

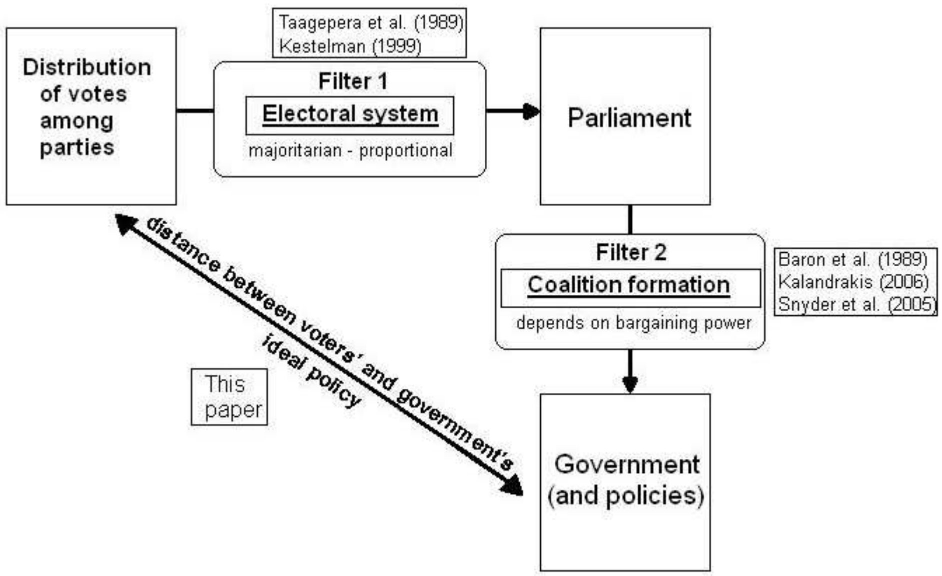

Figure 1: The government formation process

Filter 1: the electoral system. Any electoral rule but the proportional one implies some distortions.[5] Several researches have been carried out to study electoral rules's distortions.[6]

Filter 2: forming a coalition. The negotiating power of pivotal agents implies that, within the winning coalition, power is not shared proportionally to the percentage of controlled seats.[7]

[5]Proportional systems, by definition, do not distort voters' preferences: Parliament's composition perfectly reflects citizens' vote.

[6]Besides the well known paper from Taagepera and Shugart (1989),Morelli (2004) and Kestelman (1999) offer a clear and succinct review of this literature.

[7]Baron and Ferejohn (1989), Martin and Stevenson (2001), Snyder Jr., Ting, and Ansol-

A distortion at stage one is beneficial if the two distortions compensate. One can possibly imagine an infinite number of electoral rules; I restrict my attention to purely proportional and plurality rule, which are particularly common and from which most democratic countries derived their own rule.

If, after elections, a single party controls the majority of seats, it rules the country alone (filter 2 disappears in that case).[8] If no party holds a majority, a party (called 'formateur') is asked to form the government coalition that will rule the country.[9]

The probability for a party of being formateur is unobservable. Baron and Ferejohn (1989) proposes to attach to the event a probability equal to parties' share of seats in Parliament; the empirical work of Diermeier and Merlo (2004) cannot reject this hypothesis. When the formateur is successful, it belongs to the winning coalition and selects its partners. This approach, common in the theoretical literature,[10] performs well in empirical tests.[11]

The formateur forms a coalition of parties that, together, received at least half of the votes cast. Parties in the coalition should come to an agreement on the political program, the identity of the government's members and possibly the share of economic benefits. Together, they rule the country and they derive office and/or ideological benefits.[12]

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abehere (2005), Kalandrakis (2006) and Bandyopadhyay and Oak (2008) study some properties of the coalition formation game.

[8]I disregard this case. In section 3.1 I briefly discuss about that.

[9]For a detailed explanation of the role of the formateur and how it is chosen see, for instance, Diermeier and Merlo (2004).

[10]E.g., see Baron and Diermeier (2001) or Diermeier, Eraslan, and Merlo (2007)

[11]A widespread alternative, first proposed by Austen-Smith and Banks (1988), supposes that the biggest party is the first formateur. With exogenous probability it does not succeed and it is replaced by the second biggest party and so on until a party forms a winning coalition. Some authors, such as Diermeier and Roozendaal (1998), add a cost of delaying government formation to the model, to increase the probability of succeeding.

[12]Ideological benefits consist of the right to implement the preferred policy. Office benefits "could take a variety of forms [...] (including) jobs for party stalwarts, board seats on public companies [...] (In) Germany; all the major parties [...] occupy seats on the supervisory boards of the national television system and major corporations (such as Volkswagen). Moreover, each major party receives substantial amounts of public money for its research and education foundations. Similar arrangements are common in many other parliamentary democracies [...]"(Baron, Diermeier, and Fong (2007), p.8).

Part of the literature only considers office (Riker (1962), Baron and Ferejohn (1989) or Baron (1989)) or ideological (Laver and Shepsle (1990), Schofield (1986) or Baron (1993)) benefits. Austen-Smith and Banks (1990) consider both payoffs, assuming them to be orthogonal, and analyse them separately. Also in Sened (1996) both elements affect parties, which have "a utility function that amalgamates the utility functions used by Ricker and Schofield" (Sened (1996), p. 350). Martin and Stevenson (2001) empirically confirm that both elements might matter.

The total value of a coalition is given by the sum of both kinds of benefits. It is shared among members according to their bargaining power. This can happen within a cooperative or non-cooperative framework.

Cooperative coalition formation theory considers the case in which players maximise the total joint profit and then agree to share total benefit according to a rule, often called *value*.[13] The Shapley value and its modifications (e.g., Shapley-Shubik and Owen) are the most used rules.[14]

Non-cooperative game theory formalises a situation in which each player is maximising his own payoff given the best response of the other players. The literature on non-cooperative games is usually applied to frameworks in which at least some players can obtain a higher payoff by adopting selfish strategies.

In the government formation process, like in cooperative games, parties want to form a coalition (i.e., to enter the government) so as to share the payoff. Belonging to the coalition is necessary but, once reached the threshold to rule the country, there is no interest in having a bigger coalition (there is no interest, for instance, in forming the grand coalition). Moreover it is reasonable to expect that, within the coalition, all players are playing non-cooperatively.[15]

I analyse the entire electoral process, from elections to government formation; I compare purely proportional and plurality rule systems. While the previous literature (see figure 1) concentrated either on filter one or filter two, the original contribution and aim of the model is to study under which of the two systems the distribution of power among parties better reflects citizens' votes. To do that, for both electoral systems I compute the misrepresentation of voters' preferences, i.e. the Euclidean distance between the expected policy resulting from elections and voters' preferences.

I consider the three-party case and assume the number of parties to remain

[13] A *value* is a function allocating payoffs in a unique way, for each state of nature, respecting some required axioms. See Marichal, Kojadinovic, and Fujimoto (2007).

[14] Shapley-value-like rules assign to each player a power proportional to the expected value of his marginal contributions to all possible coalitions or, in the case of Shapley-Shubik, of the number of times a member is pivotal. See Shapley (1953), Owen (1977), Owen (1981), Amer, Carreras, and Gimenez (2002), van den Brink and van der Laan (2005) and Wiese (2007) for more details.

[15] In a preliminary version of this work I solved the coalition formation stage both for the cooperative and non-cooperative game and introduced a parameter α determining the degree of cooperation. Results where not changing significantly with either approach, while the description of the model mixing both became much heavier. Willing to use only one approach, I preferred the non-cooperative one because standard sharing rules needed to be adapted (I used a modified version of the Owen value) and this was distracting the reader from the main points of the paper.

unchanged when the electoral rule changes.[16] Each party can, a priori, form a winning coalition with either of the remaining two.[17]

The pivotal role of small parties might allow them to obtain more office benefits than what is justified by their share of votes. Distortive electoral rules tend to increase the power of the largest parties; if the distortion only compensate for the previously underlined distortion, distortive electoral rules can increase voters' representativeness.

Parties discount future; when parties are sufficiently patient, the majority voting system is preferable regardless of the relative share of seats: the price for the formateur of convincing other parties to form a coalition increases with patience, thus the formateur looses bargaining power. Small parties, which are rarely the formateur, obtain a large share of total benefits. For very impatient parties the preferred electoral system becomes the proportional one. Parties' relative share of seats might also matter in determining which electoral system better represents voters.

For expositional convenience, I consider that the winning coalition is sharing a budget and parties, being self-interested, only care about their share of the budget.[18]

The paper is organised as follows: section 2 describes the model and its assumptions. Section 3 illustrates results and discusses (subsection 3.1) the consequences of relaxing some of the initial assumptions. Section 4 illustrates the model using the Italian data from 2006 Senate and 2008 House of Representatives elections. The last section concludes.

2 The model

The political process follows (in representative democracies and in my model) the following chronological scheme:

1. Elections take place

[16]Having a fixed number of parties can be considered as a short term assumption. I leave for future researches the study of long term effects of a change in the electoral rule in the n-party.

[17]In the literature on spatial competition among parties, some authors (e.g. Axelrod (1970) or de Swaan (1973)) introduce the impossibility for some parties, for a priori ideological reasons, to form a coalition together. In my model this would not add any extra insight.

[18]Alternatively, the reader could think a) of different projects to be financed and each party is interested in only one of them or b) that production of laws is time consuming and parties should fix the time to devote to each of them to legislate.

2. Given the electoral rule, from votes we obtain the share of seats attributed to each party
3. A randomly chosen formateur tries to form a coalition; negotiation begins
4. A formateur succeeds when a majority coalition is formed and the government is chosen. In case of failure, again a formateur is randomly selected and negotiations start again
5. When a coalition is formed, benefits are shared within its members

I consider a country with $n = 3$ groups of homogeneous citizens indexed by i. The relative size of each group is denoted by c_i. Without loss of generality, I order groups by their size, thus $c_1 > c_2 > c_3$.

For each group, the political program of the corresponding party i maximises the utility of the group. Vector e regroups the share of seats that parties obtain after the election. An electoral system is seen as a function F transforming parties' share of votes into shares of seats, i.e., $e = F(c)$. I focus on two systems: proportional and majority voting (also called "plurality rule").[19]

Assumption 1 (No standing-alone) *No party obtains the majority of seats, neither under a proportional electoral rule nor under plurality rule. This means that* $e_1 < 0.5$ *and, a fortiori, that* $c_1 < 0.5$*.*

Figure 2 shows the possible combinations of e_2 and e_3 respecting the ordering $0.5 > e_1 > e_2 > e_3 > 0$.

Assumption 2 (Constant value of the coalition) *I assume that the total amount of resources to allocate is constant. Consequently, the bargaining issue boils down to the classical "sharing a dollar" problem, where each party (and its voters) is only interested in its share of total budget.*

When no party secures a majority, the bargaining phase begins. A coalition S is the result of an agreement between two parties on how to share the budget. Let Z denote the set of all feasible allocations (i.e., $Z = \{z \in R^3_+ : \sum_1^3 z_i \leq 1\}$), z_i is the budget share of party i. A winning coalition has to be supported by at least

[19] By proportional rule I mean that the share of seats in Parliament is equal to the share of votes received. By plurality rule I mean that the country is divided in districts (as many as the number of available seats) and in each district the candidate who receives the largest share of votes wins.

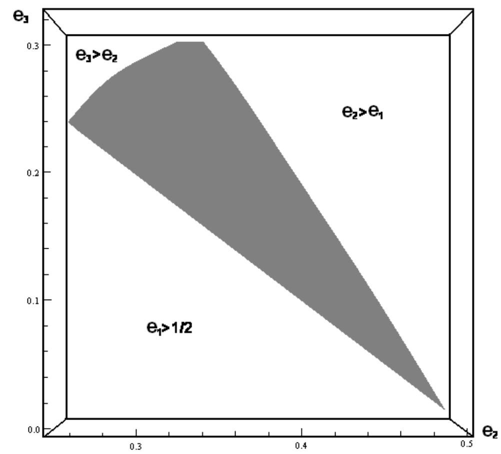

Figure 2: Acceptable combinations of e_2 and e_3.

half of Parliamentarians. Agents' utility is linear in z_i and do not depend on z_j (i.e., $U_i(z) = z_i$). The set $D \subseteq 2^n = 8$ is the set of all possible winning coalitions.

To model the bargaining game, I follow Kalandrakis (2006) which proposes a game à la Rubinstein-Ståhl (Rubinstein (1982)). At time $t = 0$ a party, called formateur, is randomly chosen.

Assumption 3 (Recognition probability) *The recognition probability π_i of being a successful formateur is equal to the share of seats that party i controls (i.e., $\pi_i = e_i$).*

The formateur should form a coalition $S \subset D$, which means it has to propose a vector z which should be approved by the parties in S.[20] If z is accepted by S the game end: the government is formed and the budget is shared according to z. Otherwise, in the next period ($t = 1$) a formateur (possibly the same one) is randomly chosen and the game continues until an agreement is reached. Henceforth I will use the notation z_j^i to indicate the j^{th} element of vector z when i is the formateur.

Assumption 4 *All parties discount future at the same rate $\delta < 1$.*

Because of the discount factor, the utility in time t of a share z_i in period $t + k$ is given by $U_i(z, t, t + k) = \delta^k z_i$. Before knowing the identity of the formateur, the

[20]To have a winning coalition, all parties in S should obtain a positive share. There is no reason to leave a positive share to parties that do not belong to the coalition.

expected utility of a party depends also on the probability that a given coalition forms: $EU_i(z, t, t + k) = \delta^k \sum_{h \in D} \pi_h z_i^h$. Parties' outside opportunity is zero.

Given the vector of seats share e, the time discount factor δ and the set of winning coalitions D, a game with 3 parties is denoted by $\Gamma(3, D, \delta, e)$. When an agent is formateur, its action consists in proposing a division $z^i \in Z$ of the budget, the others' action space consists in the choice to accept or refuse the formateur proposal. I focus only on stationary proposal strategies involving no delay: in each period a party always behaves the same way when formateur and, without delay, all parties belonging to the proposed coalition accept. The continuation value, v, is defined as the vector of the expected utilities of parties if we move to the next period.

Party i's continuation value v_i is the expected utility of i if the game moves in the next period. A no-delay, Stationary, Subgame Perfect, Pure Strategy (SSPPS) Nash equilibrium for game $\Gamma(3, D, \delta, e)$ is a set z^i of stationary strategies, to which corresponds a vector v of continuation values such that $v_i = \sum_{h \in 3} \pi_h z_i^h$ (with $z_i \geq \delta v_i$) for all $i \in 3$.[21] Existence of SSP Nash equilibrium is not an issue for game $\Gamma(3, D, \delta, e)$ by the arguments of Banks and Duggan (2000). SSPPS equilibria in this game are both Nash and Subgame Perfect. Other equilibria exist, but I only concentrate on the stable pure strategy ones.[22]

Measuring misrepresentation. The ultimate goal of the model is to relate voters' preferences[23] to government policies[24]. We have all the information to compute (given voters' choice) the winning coalition and the relative parties' share. So as to determine which electoral system leads to the share of power closer to voters' distribution of preferences, I measure misrepresentation through M^y, where $y = \{PR; MV\}$ denotes the electoral system, with PR=proportional rule and MV=majority voting.

[21]For more details on that, see Kalandrakis (2006), p. 444.

[22](Snyder Jr., Ting, and Ansolabehere (2005), for the same model, finds one of the mixed strategies equilibria).

[23]Voters' preferences are inferred from their vote. As previously announced, voters are supposed to vote sincerely. Even though it is not always the case, this event is not empirically irrelevant.

For instance, Hooghe, Maddens, and Noppe (2006) found empirical evidence that after the last voting rule change in Belgium there were no significant changes in voters' behaviour and they interpreted it as a signal of myopic/sincere voting.

[24]By government policies I mean the amount of resources devoted to each project.

$$M^y = \sqrt{\sum\nolimits_{i=1}^{n} \left(\sum_{S \in N} \Pr(S) \cdot z_i^S - c_i \right)^2} \qquad (1)$$

$\Pr(S)$ is the probability for coalition S to be formed and thus of a formateur to be successful. Note that the electoral system determines the number of seats a party controls and thus it possibly affects both the probability of a coalition forming and each party's budget share.

Equation (1) represents the sum of the Euclidean distances between parties' expected budget share and the optimal one. This is the square of the difference between parties' expected power and their proportion of supporters (c_i). Indeed, equation 1 computes the expected share of benefit (discounted for the probability of forming each possible coalition) and then it computes the Euclidean distance between this point and the ideal point where each party's power is precisely equal to its share of votes.

Equation (1) takes large values when parties are either under-represented or over-represented. Over-representation occurs when a party, being pivotal for a coalition, obtains a larger share of benefits than the share of population it represents.

To compute M^y, one needs to know both c_i and e_i (necessary to compute $\Pr\{S\} \cdot z_i^S$). The relation between the two variables depends on the electoral system. Taagepera and Shugart (1989) show that virtually every electoral system can be approximated through a function $e_i = F(c_i) = \frac{c_i^\tau}{\sum_{j=1}^{n} c_j^\tau}$ by properly choosing the value of the parameter τ. By construction, for the proportional system $\tau = 1$, i.e. $c_i = e_i$. Under plurality rule, the share of seats depends on the geographical distribution of voters' preferences over districts. For plurality single-member districts systems, it is usually considered that $\tau \approx 3$ (Qualter (1968)) and for that was named the 'cube rule'.[25] According to Taagepera and Shugart (1989), $\tau = 2.5$ is more appropriated than $\tau = 3$ for modern western societies with plurality single-member district systems while $\tau = 8$ would be a better approximation for the USA actual system. In the literature, the original cube rule (with $\tau = 3$) is usually assumed to be a good approximation for the two party case and also to fit data for the three party case; when the number of parties increases the precision of this measure falls.

[25]This relation was originally formulated by J. P. Smith in a report to the British Royal Commission on electoral systems in 1909 and then developed and made famous by Duverger (1954). For a discussion on it and its drawback, see for example Riker (1982), Rogowski and Kayser (2002) or Blau (2001).

Since in this model we are only interested in the number of seats each party obtains, I will not make any assumption on the district distribution of preferences and I rather assume that the *cube rule* holds, with $\tau = 3$.[26]

Assumption 5 (Cube rule) *To compute the share of seats of a party under majority voting given the share of votes received, I assume that we observe the aggregate number of votes for each party (c_i) and the cube rule holds, thus* $e_i = \frac{c_i^3}{\sum_{j=1}^3 c_j^3}$.

3 Model results

I begin with the analysis of how coalitions are formed (I show that it is always preferable to form a coalition with the smallest other party) and compute the z vector. Then I compare the misrepresentation of preferences (equation (1)) under proportional and majoritarian systems, concluding that, for some values of δ, under proportional voting systems there is a greater distortion of voters' preferences than under plurality rule.

With three parties, a coalition of two of them is always sufficient to form a majority. The formateur chooses the other member of the coalition by comparing its utility in each possible coalition. Ex ante, $2^N = 8$ different scenarios might occur depending on the identity of the formateur.

Proposition 1 (Sharing rule) *Denoting i the formateur, it proposes to party j its continuation value and zero to the other one. In particular, the share vector* $z^i = \left(z_i^i; z_j^i; z_x^i\right)$ *takes the following values* $\left(1 - z_j^i; \frac{\delta}{1-\delta e_i}(e_j z_j^j + e_x z_j^x); 0\right)$. *Party x is excluded from the coalition and receives zero, while party j obtains the present value of what it would get (in discounted expected terms) in the next period.*

Proof. There is no reason to form a coalition with more than one other party, since with three parties any coalition of two of them is sufficient to control the majority of seats. The cheapest price to induce a party to accept a proposal is to offer its next period discounted profit. Its expected profit depends on the expected identity of the formateur in the next period and on which coalition would thus result. ■

Proposition 2 (Minimal winning coalition) *In the SSPPS equilibrium, the formateur always chooses to form a coalition with the smallest of the two other*

[26]Of course all my results can be replicated with different values of τ to better fit a specific country electoral system and geographical distribution of preferences among voters. See section 4.1 for more on that.

parties.[27] *The ex-ante unique equilibrium scenario is the one in which the following coalitions can form:* {*1,3*},{*2,3*} *and* {*3,2*}.[28]

Proof. In the appendix. ■

Corollary 2.1 (Probability of forming a coalition) *Given the probability of being the formateur, the probability of forming coalition* $\{i,j\}$ *are the following:* $Pr(\{1,3\}) = e_1$, $Pr(\{2,3\}) = e_2$ *and* $Pr(\{3,2\}) = e_3$.

Corollary 2.2 *Equilibrium parties' shares* z *are summarised in table 1.*

$z^1 = \left(\frac{(1-\delta)(1-\delta e_3)}{1-\delta+\delta^2 e_1 e_3}; 0; \frac{(1-\delta e_2-\delta e_3)\delta e_3}{1-\delta+\delta^2 e_1 e_3}\right)$
$z^2 = \left(0; \frac{(1-\delta)(1-\delta e_3)}{1-\delta+\delta^2 e_1 e_3}; \frac{(1-\delta e_2-\delta e_3)\delta e_3}{1-\delta+\delta^2 e_1 e_3}\right)$
$z^3 = \left(0; \frac{(1-\delta)\delta e_2}{1-\delta+\delta^2 e_1 e_3}; \frac{(1-\delta)(1-\delta e_2)+\delta^2 e_1 e_3}{1-\delta+\delta^2 e_1 e_3}\right)$

Table 1: The equilibrium vectors z.

Proposition 2 can be easily interpreted. Small parties are "cheaper to convince to form a coalition", thus the formateur always tries to form a coalition with the smallest available party. From table 1, one can notice that the discount factor δ plays a key role in the budget share. Since the formateur is residual claimant, it always pays to its partner its discounted continuation value. The more patient are the other players, the larger this value. When δ is close to one (parties are patient), the formateur is forced to let almost all the share to the other party.

This result differs from the one in Snyder Jr., Ting, and Ansolabehere (2005), which suggests that players' equilibrium shares are the same, because I concentrate on the pure strategy equilibrium while Snyder Jr., Ting, and Ansolabehere (2005) show results from one of the mixed strategies equilibria (with parties competing on price to belong to the winning coalition, the formateur can extract more surplus from them and also that prices are the same for all parties).

[27] This result is in line with the empirical evidence that parties try to form coalition with the smallest parties and to have minimal winning coalitions.

The reason is clear: it is hard to coordinate a large coalition and government's efficiency decreases. Parliamentarians in the coalition try each to influence government's policy to pursue their own agenda and coordination costs increase in the size of the coalition. Both theoretical (i.e., Gamson (1961), Leiserson (1968) and Riker (1962)) and empirical (i.e., Martin and Stevenson (2001)) works confirm that the formateur tries to form a minimal winning coalition. In Auriol and Gary-Bobo (2008) there is a relatively close result: the bigger the size of Parliament (and similarly of the winning coalition), the more costly it is both to find agreements and to avoid being (partially) captured or lobbied.

[28] Which one occurs depends on the identity of the formateur, thus on the share of votes each party received.

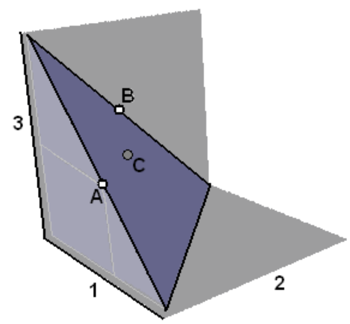

Figure 3: Coalitional space Z

Figure 3 shows all possible combinations of budget share amongst three parties. On the axis are parties' budget shares. The dark side of the simplex is the set Z. Points A and B are examples of possible budget shares when a coalition forms between respectively parties 1 and 3 and between 2 and 3. Point C is an example of where the optimal[29] point might be. While point C does not depend on δ, points A and B do. The larger the value of δ, the further points A and B are from the formateur axe.

By proposition 1, the equilibrium share always lies on a vertex (a party ends up with nothing). Note that the equilibrium share depends on the identity of the formateur; thus, *a priori*, $z_j^i \neq z_i^j$.

Combining the information in table 1 and equation 1 it is possible to compute the misrepresentation of voters' preferences under the proportional rule (M^{PR}) and under majority voting (M^{MV}).

Equation 1 can be rewritten as:

$$M^y = \left[\left(e_1 \frac{(1-\delta)(1-\delta e_3)}{1-\delta+\delta^2 e_1 e_3} - c_1\right)^2 + \right. \tag{2a}$$

$$\left(e_2 \frac{(1-\delta)(1-\delta e_3)}{1-\delta+\delta^2 e_1 e_3} + e_3 \frac{(1-\delta)\,\delta e_2}{1-\delta+\delta^2 e_1 e_3} - c_2\right)^2 + \tag{2b}$$

$$\left.\left(\frac{(e_1+e_2)(1-\delta e_2-\delta e_3)\,\delta e_3}{1-\delta+\delta^2 e_1 e_3} + \frac{e_3(1-\delta)(1-\delta e_2)+\delta^2 e_1 e_3}{1-\delta+\delta^2 e_1 e_3} - c_3\right)^2\right]^{0.5} \tag{2c}$$

where 2a, 2b and 2c are the distances, respectively for party 1, 2 and 3, between

[29]Optimal in the sense that each party obtains a share of the budget equal to the share of citizens supporting it.

the expected budget share and the share of voters supporting each party.

Computing misrepresentation under proportional rule is straightforward; seats and votes shares are the same. Equation (2) becomes

$$M^{PR} = \frac{1}{\left[1-\delta+\delta^2 c_1 c_3\right]}\left[\left(\left(1+\delta\right)\delta c_1 c_3\right)^2 + \delta^2 c_1^2\left(\left(\delta c_1 - 1 - \delta\right)c_3 - 2\right)^2 + \left(\delta c_2 c_3\left(2+\delta c_1 - 2\delta\right)\right)^2\right]^{0.5} \tag{3}$$

Using the cube rule (assumption 5) to obtain the relation between c_i and e_i, equation (2) becomes

$$M^{MV} = \left[\left[c_3^3\left[\left(\delta-1\right)\left[\left(c_3^3+c_2^3\right)c_3^3\delta+\sigma^2\right]-\delta c_3^6\left(1+\delta\right)\right]-c_3\sigma x\right]^2 + \left[c_2^3\left[\left(1-\delta\right)\sigma^2-2c_3^6\delta^2\right]-c_2\sigma x\right]^2 + \left[c_3^3\left[c_1^3\sigma\delta+2\left(c_1^3+c_2^3\right)c_3^3\delta^2+\left(1-\delta\right)\sigma^2\right]-c_3 x\sigma\right]^2\right]^{0.5}\frac{1}{\sigma x} \tag{4}$$

where $\sigma = \sum_{i=1}^{3} c_i^3$ and $x = \left(1-\delta\right)\sigma^2 + c_1^3 c_3^3 \delta^2$.

It is now possible to compute the difference in misrepresentation between the two electoral systems $MM = M^{PR} - M^{MV}$:

$$MM = \frac{1}{\left[1-\delta+\delta^2 c_1 c_3\right]^2}\left[\left(\left(1+\delta\right)\delta c_1 c_3\right)^2 + \delta^2 c_1^2\left(\left(\delta c_1 - 1 - \delta\right)c_3 - 2\right)^2 + \left(\delta c_2 c_3\left(2+\delta c_1 - 2\delta\right)\right)^2\right]^{0.5} - \left[\left[c_3^3\left[\left(\delta-1\right)\left[\left(c_3^3+c_2^3\right)c_3^3\delta+\sigma^2\right]-\delta c_3^6\left(1+\delta\right)\right]-c_3\sigma x\right]^2 + \left[c_3^3\left[c_1^3\sigma\delta+2\left(c_1^3+c_2^3\right)c_3^3\delta^2+\left(1-\delta\right)\sigma^2\right]-c_3 x\sigma\right]^2\right]^{0.5}\left(\frac{1}{\sigma x}\right)^2 \tag{5}$$

When equation (5) is positive, plurality rule represents voters' preferences better than the proportional rule (i.e., each party's expected budget share is closer to the number of votes the party received). In fact, $MM > 0$ means that the difference between parties' expected and optimal share is greater under the proportional than the plurality rule.

Proposition 3 (Role of the discount factor) *Two thresholds* $(\underline{\delta};\bar{\delta})$ *exist for the discount factor such that, for any combination of seats shares between parties:*

a) majority voting is preferable when $\delta > \bar{\delta}$

b) proportional rule is preferable when $\delta < \underline{\delta}$.

When the value of δ *is within the two thresholds, the relative share of seats among parties determines whether the majority voting system is preferable or not. The two thresholds for* δ *are* (10.8%; 78%).[30]

Proof. This can be proved either through numerical simulation or graphically. Figure 4 shows how equation 5 changes, for six levels of δ. The horizontal axe depicts the share of seats of party 3 (e_3), while e_2 is given by the depth axe. ■

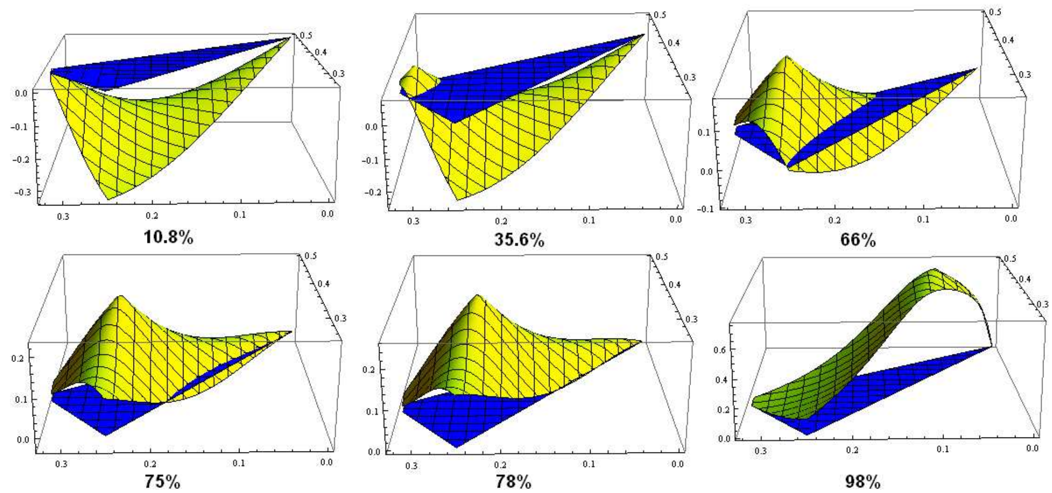

Figure 4: The impact of δ

Outside the interval $\delta \in (10.8\%; 78\%)$ misrepresentation depends only on δ and either electoral system is always preferred. On the opposite, within that interval parties' relative share of seats also matters.

Proposition 3 can be easily interpreted. After elections a coalition forms: the formateur lets its partner a share of budget at least equal to the expected share the partner would otherwise obtain in the next period. When δ is small enough (parties are impatient), it is cheap to persuade a partner: indeed when δ tends to zero, the formateur's share of budget tends to one and parties' expected utility tends to their share of seats. Thus, the best electoral system is the one for which

[30] The value for the two thresholds depends on the number of parties; 10.8% and 78% are the values for the three-party case. The remaining of the proposition holds even with a larger number of parties.

$c_i = e_i$, i.e. the proportional one.[31] In other words: filter 2 (picture 1) disappears for δ going to zero and there is no reason to distort the mechanism at filter-one level.

The share the formateur has to leave equalises its partner's discounted expected earning. When δ gets larger (parties are patient), a distortion appears at filter-two level since the portion increases of future earnings the formateur has to leave to its partner; the expected share of small parties becomes larger than the share of votes they received. The majority voting electoral rule distorts election results in the opposite direction (reducing the share of seats and of budget of small parties); when δ is large enough ($\delta > 78\%$), the distortion level is large and the majority rule desirable. When $\delta \in (10.8\%; 78\%)$, the discount factor guarantees that the excessive share of small parties is quite small: the majority rule distortion then might be too large. When the smaller party is very small the majority rule distortion is larger than what necessary to balance the coalition formation distortion (i.e., the distortion at filter 1 level induced by plurality rule is too large with respect to the one at filter 2 level). For $\delta \in (10.8\%; 78\%)$ and c_3 close to zero the proportional rule is preferable while, for c_3 and c_2 big enough, majority voting is better than proportional rule.

3.1 Relaxing some assumptions

Before moving to next section, I briefly describe how model's results would change relaxing some assumptions. The most delicate assumption is probably that the number of parties is limited to three. This is a simplifying assumption that allows to obtain some closed form results. It is possible to solve the model with a larger number of parties but it would be necessary, to have clear results, to impose some restrictions on the relative share of votes of parties. The first impact of having more parties comes from the choice of partners. Namely, proposition 2 might not hold, forming a coalition with the smallest party might not be enough to control the majority of seats and parties might try to minimise the total size of the coalition both in terms of represented parties and number of controlled seats. According to the kind of coalition that is formed, the thresholds for δ would change but qualitatively results would be the same.

[31]From an analytical point of view: $\delta \to 0$ implies $z_j^i \to 0$, thus $z_i^i \to 1$ and $EU_i(z_i^S, t, t) = \sum_{h \in D} \pi_h z_i^h \to \pi_i$. Since $\pi_i = e_i$, to minimise the difference between the share of votes (c_i) and the expected share of budget for a party ($EU_i(z_i^S, t, t)$) we need $e_i = c_i$, which is the case under the proportional electoral system.

Relaxing assumption 1 can lead to two scenarios: if a party obtains the majority of seats regardless of the electoral system, then any electoral rule is equally representative of voters' preferences. Additional, non trivial, researches are required for the case in which one party controls the majority alone only under a regime.

When allowing (contrary to assumption 2) coalitions to have different values (for instance, this would be true if a party has ideological affinities with another), coalition formation would integrate this. The same considerations as for the number of parties would hold. I excluded this case, because results would crucially depend on the hypothesis to model parties' affinity.

The last consideration is above the common value of δ (assumption 4). What determines parties' share is the discount factor of the non-formateur party. All results can be thus extended just by replacing δ with the discount factor of the party belonging to the winning coalition other than the formateur.

4 A model illustration with Italian elections' data

In the introduction I claimed that in Italy, over the two last legislatures, the smallest parties belonging to the winning coalitions had very much power given their seats share in Parliament. In this section I use the official results of the Italian elections to instance the model.[32]

By the Italian law, Parliament is divided into two houses: the lower house and the Senate. All adults (older than 18 years old) can vote for the first house, while only citizens aged more than 25 years old can vote for the Senate (this explains some minor differences in the number of votes parties receive in the two houses). The most recent electoral law implies different electoral rules for each house and includes some regional differences too. Over the 20 Italian regions, in 18 (19 for the congress) the electoral law is based on the proportional principle, but with local distortions, possibly different in each region. For my computations, I used the number of votes a party received and not its number of seats, in order to disregard the Italian distortions.

In 2006 a centre-left coalition elected Romano Prodi prime minister. Even though the winning coalition included twelve parties (eight at the senate and

[32]The official election results are public and can be easily found on the Italian government web page, as well as in the specific web page of each of the two houses of Parliament, on the web page of the *Ministero degli interni* and in many independent web pages.

eight at the lower house), many of these parties were just created *ad hoc* before the election to take advantage of some peculiar mechanisms of the non standard electoral law and only few of them were real independent parties, with an own agenda (different from the one of the main party in the coalition) and some strong personalities managing them. In the Senate, the smallest independent party in the coalition was UDEUR. On a national basis it represented about 1% of citizens. UDEUR was pivotal and, when in 2008 it decided not to support Mr. Prodi anymore the government lost the majority and new elections were called.

Coalition members were aware of the consequences of one party leaving. Including the support given to Mr. Prodi by some "senatori a vita"[33], the majority controlled three more senators than the opposite coalition. During the last months of Mr. Prodi's government, the three UDEUR's senators and a few other senators in the coalition (e.g. Mr. Dini) used their influence on medias and their pivotal positions (threatening to leave) to obtain some major changes in several law proposals and especially in the "Finanziaria".[34] They carried on their own agenda and clearly showed that their power within the coalition was much higher than the 1% they were supposed to represent.

In what follows, I illustrate my model considering the left and the right coalitions as two independent parties (named CL and CR) and UDEUR as a third independent party that can form a coalition with either coalition.[35] The first column of table 2 summarises the 2006 Italian Senate share of seats under proportional representation. The second column is an estimation (using the "cube rule") of the number of seats each party would have obtained with a majoritarian electoral system.

	Seats - Proportional Rule	Seats - Plurality Rule
CL	49.5%	49.999%
CR	49.5%	49.999%
UDEUR	1%	0.002%

Table 2: Seats Share - 2006 Italian Senate

The share of budget depends on the discount factor. Picture 5 depicts the

[33] "Senatori a vita" are senators who are not elected (such as former Republic Presidents). They sit in Parliament for life.

[34] The "Legge Finanziaria" is one of the most important Italian laws, voted every year to determine the way public budget is used during the forthcoming year.

[35] This would have been plausible even from the political point of view, since it is a centre (catholic) party and its chief in the past already accepted to be in some coalitions with some centre-right parties.

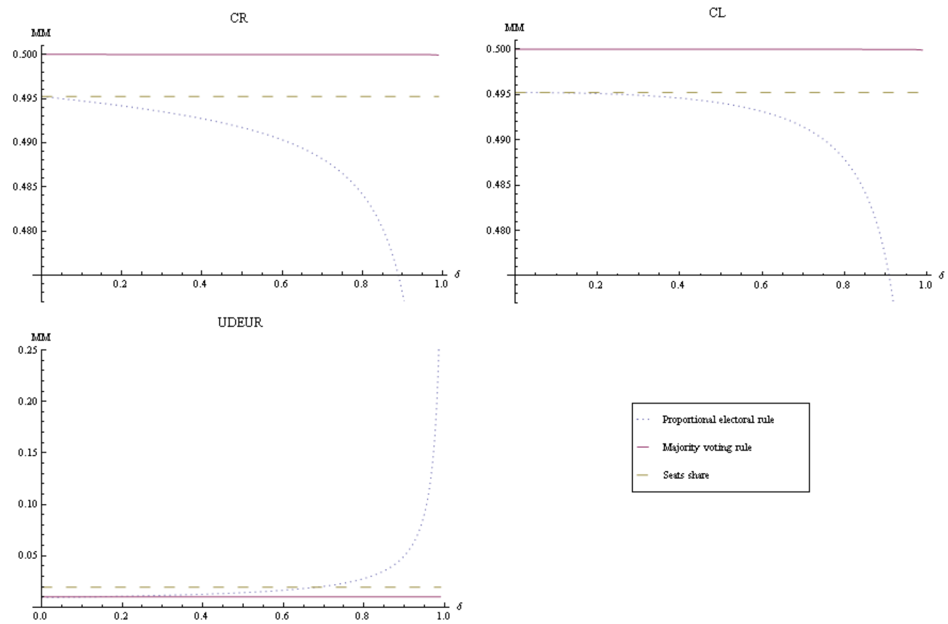

Figure 5: Budget share depending on δ

equilibrium share of each party for different levels of δ. The violet straight line is for the majority voting case, while the blue dotted one is for the proportional rule and the yellow dashed one represents the share of votes each party obtained at the election. The top picture shows the share for CR, the central one for CL and the bottom one for UDEUR. As you can notice, with the majority voting system the share of both CR and CL increases at the expenses of UDEUR. Observe also that UDEUR's share can be extremely large under the proportional rule when the discount factor tends to one (e.g., for $\delta = 0.99$, under proportional representation its expected share is 33%, while under plurality rule it is 0.02%).

Equation (1) measure the Euclidean distance between the average power of a party and the optimal one, given voters preferences. The larger its value, the bigger is the difference between the distribution of power in the government and the distribution of voters' preferences.

With data on the 2006 Italian elections, the difference in misrepresentation (MM, equation 5) is drawn in picture 6 as a function of δ. For $\delta > 66.7\%$, the majority voting rule ensures a better representation of voters, while the opposite is true for $\delta < 66.7\%$.

Similarly to what happened in 2006, after the 2008 elections a coalition has been formed. Due to a change in the political strategy of the two main parties, only

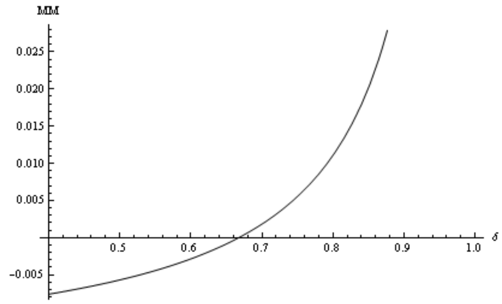

Figure 6: Misrepresentation as a function of δ

four parties are now represented in Parliament: PD-IDV (the centre-left party)[36], PDL (the centre-right party), Lega Nord and UDC, the smallest party.[37]

	Seats - Proportional Rule	Seats - Plurality Rule
PD-IDV	41.7%	49.8%
PDL	41.5%	49.2%
Lega Nord	10.5%	0.8%
UDC	6.2%	0.2%

Table 3: Seats Share - 2008 Italian Congress

UDC did not obtain enough seats to form a two party coalition (see table 3) and resulted to be a "dummy player", that is: regardless of the coalition, its contribution is always irrelevant, thus it never belongs to a winning coalition.

This time the successful coalition was the centre-right one (PDL with Lega Nord), and Mr. Berlusconi was elected prime minister. Even though the new coalition is in power only since a few months, Lega Nord has already proved that it is not willing to accept the coalition's decisions without negotiating. PDL already lost more than once the support of Lega Nord on some law proposals that were not in line with Lega Nord's program.

Figure 7 again shows the share of budget of each party, according to the value of δ for PD (top), PDL (centre) and Lega Nord (bottom). Again, under the

[36]Before the elections, PD signed a pre-electoral agreement to run with IDV. They agreed on the program and the candidate prime minister. Because of the electoral rule, they considered it would have been more convenient to keep the two different party names.

[37]To be more precise, one more party (SVP) is represented. SVP is a local party from a cross-border region (close to Austria), where the majority of citizens speaks German. They received, on a national base, 0.5% of votes. To contribute to the defence of linguistic minorities, they have a special electoral legislation that allowed this party to obtain 2 seats at Parliament (equivalent to 0.3%). I did not consider votes to those parties who were not represented in Parliament because too small.

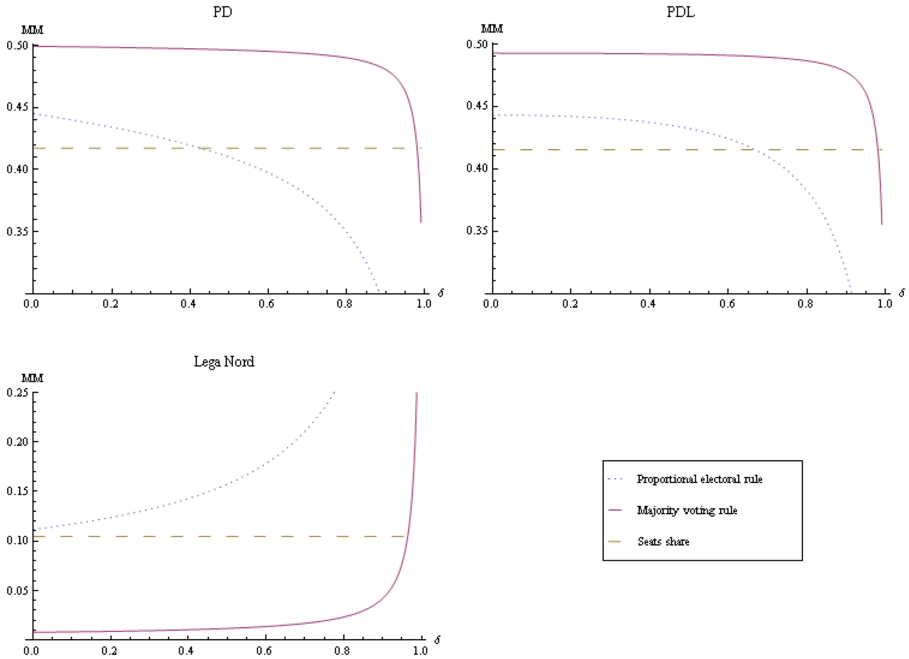

Figure 7: Budget share depending on δ

proportional rule the smallest party obtains a share of budget (for $\delta > 27\%$) considerably larger than the share of votes received (e.g., with a share of votes of 10.5% and for $\delta = 80\%$ the expected share of budget for Lega Nord is 26.8% while under majority voting the share would be 2.5%).

Considering aggregate data and the level of misrepresentation computed by equation 5, for this example we obtain the result drawn in picture 8, where we can see that the threshold for δ is $\delta = 64.64\%$ and above this threshold the majoritarian rule is preferable to the proportional one.

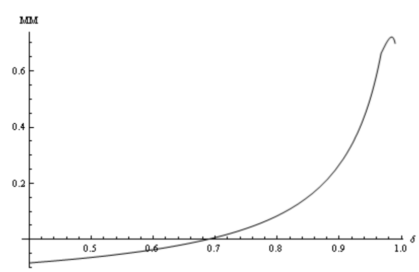

Figure 8: Misrepresentation as a function of δ

4.1 Some comments on the cube rule

Up to now, I assumed the cube rule to hold. The cube rule is an empirically tested measure. In fact, the real share of seats depends on the distribution of preferences over districts.[38] As previously suggested, it might be that for some countries $\tau = 3$ is not the best proxy one can use and different values for τ can take into account for idiosyncratic situations (different electoral system, distribution of voter preferences...).[39]

Let's see what happens to last section's results when letting both δ and τ varying. Concerning the two previous examples, in the first case (2006 Senate) we have two big parties and one very small; in the second case, even the smallest party is relatively big.

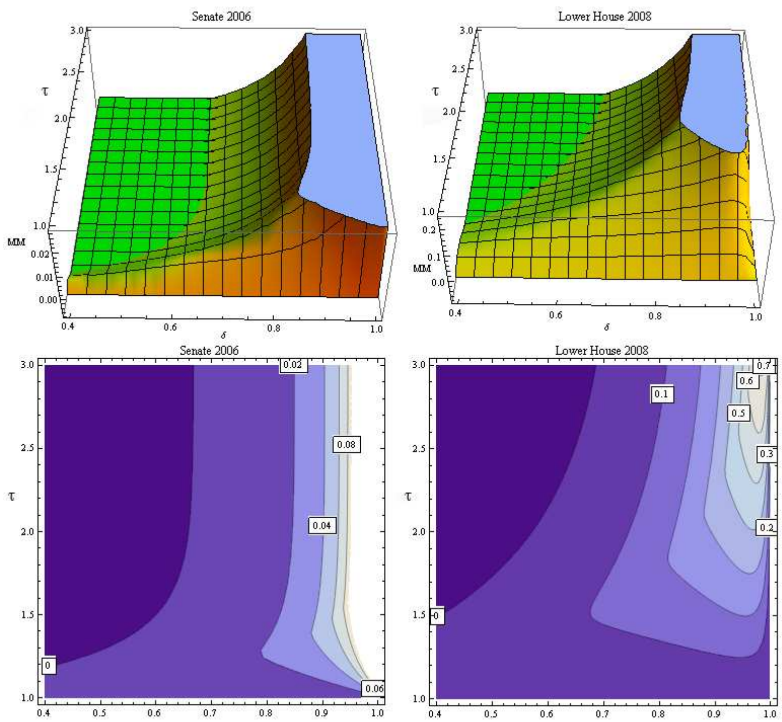

Figure 9: Changes both in δ and τ

[38]In a country, for instance, with three parties obtaining in each district the following share of votes (40%, 30%, 30%), with the majority voting rule the first party obtains 100% of seats while the cube rule would predict a share (54%, 23%, 23%).

[39]It is hard to forecast (without an empirical study) if for a given country the value of τ should be greater or smaller than three. As a general rule, we should expect that in a country in which one big party compete in all districts versus small local parties τ should be larger, while for countries with very heterogeneous districts, strong local parties and no big national parties the value of τ should be smaller than three.

Picture 9 shows how the misrepresentation index changes not only over δ but also over τ. We can notice that whatever the value for τ, non proportional voting systems perform better if parties are patient.

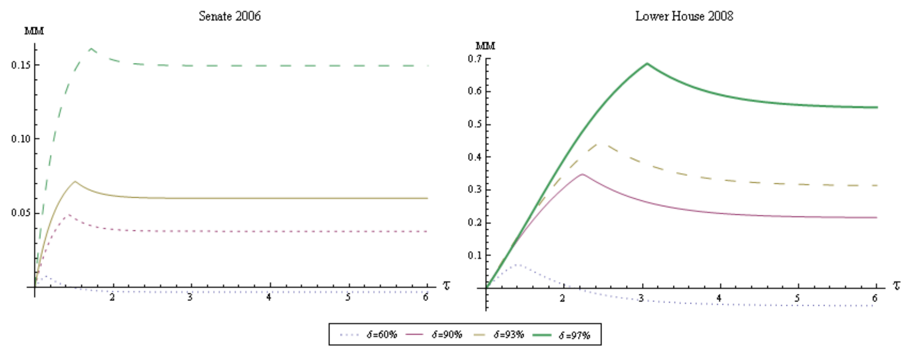

Figure 10: Effect of τ for different levels of δ

For low levels of δ we notice that non proportional systems perform better only when the value of τ is slightly larger than one. This is clearer in picture 10, where we can see that the peak can be very close to one (each line corresponds to a different level of δ, lower lines are for lower values of δ). When the smallest party is very small, the majoritarian rule might distort too much and thus it is preferable to have a small value for τ (this can be obtained artificially, with a mixed electoral rule, or it can simply be a consequence of the geographical distribution of preferences: for instance, it might be that in each district there is only one very strong party, possibly obtaining the totality of preferences). On the opposite, when the smallest party obtains a large share of votes, the majority voting distortion is smaller and it is preferable to have a τ closer to three.

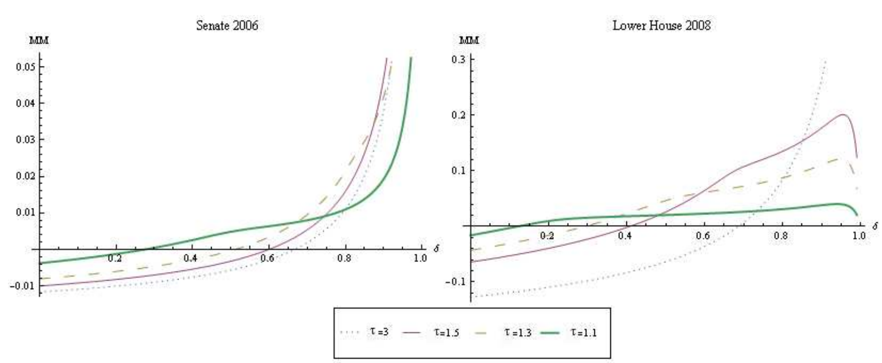

Figure 11: Effect of δ for different levels of τ

Picture 11 shows how the level of misrepresentation changes with δ for different levels of τ. Note that for the value of δ for which proportional and majority voting rule are equivalent is an increasing function of τ, that is: the smaller the value of τ, the more we are sure that (regardless of the time discount factor) introducing some elements of distortion to a proportional system is beneficial. Nevertheless, when parties are very patient it is better to have τ larger.

5 Conclusions

Electoral systems are always a social compromise balancing different interests. Each country chooses its voting rules according to political, cultural, historical and social reasons. Many countries in Europe, for instance Italy, show a preference towards proportional systems while others countries (e.g. U.K. or U.S.A.) have chosen a majority voting system.

Most countries have adapted their system according to local needs. My work has considered only the two basic electoral systems (i.e. purely proportional versus plurality rule) disregarding all the local specificities.

Proportional electoral rules are costly in term of governability: the number of represented parties tends to increase both in Parliament and in the winning coalition. The expected duration of governments falls and the average time to introduce structural changes increases because of the long negotiation time needed to find an agreement.

According to supporters of proportional systems, this is the price to pay to ensure that decisions reflect citizens' preferences. It is clear that through a proportional system, by definition, Parliament's composition reflects precisely the distribution of preferences over the country.

It is generally disregarded that decisions are mainly taken by the government and that within Parliament decisions are normally taken by the majority of members. Coalitions form to support a government and the real power of a party is determined by his role in the coalition and not by its number of seats in Parliament. Given the distortion due to negotiation and the importance of bargaining during the coalition formation stage, it results pointless to measure the degree of representativeness of Parliament. What really matters is the relation between voters' preferences and a party's power within the government.

My work consisted in showing that when parties are sufficiently patient at the coalition formation stage, the distortion derived by the negotiation process (filter 2) increases small parties' power. At the election stage (filter 1) plurality rule distorts Parliament's representativeness (contrary to the proportional rule) and the two distortions have opposite sign. If parties are very impatient, filter 2 distortion is negligible and thus a non-distorting electoral system is better, but when parties are patient enough, the magnitude of distortion increases and it becomes beneficial to introduce some distortions.

My model shows that the idea that proportional electoral rules induce government to better represent citizens' preferences (with respect to plurality rules) is false and a majority voting rule can be preferable from a representativeness perspective.

The Italian example is instructive: during the 2006-2008 government UDEUR, a party representing 1% of citizens, had the power to threaten the government, to obtain the change of many elements in the 2008 "Finanziaria" law and to determine the government's fall the 23rd of January 2008. Similarly Lega Nord, with about 11% of votes at the 2008 elections, during the six first months of the 16th Italian legislature has voted several times against the laws proposed by the government to which it belongs and obtained some major changes in some very discussed laws, such as those concerning the justice reform, the immigration laws and the law on federalism. With a less proportional system (and for instance under plurality rule), small parties' role would reduce and more decisions would be taken by parties representing a larger subset of the population.

Appendix

A Proof of proposition 2

The generic share of budget with three parties is resumed in table 4.[40]

	Formateur		
Shares	1	2	3
z_1	$1 - z_2^1 - z_3^1$	$\frac{\delta}{1-\delta e_2}\left(e_1 z_1^1 - e_3 z_1^3\right)$	$\frac{\delta}{1-\delta e_3}\left(e_1 z_1^1 - e_2 z_1^2\right)$
z_2	$\frac{\delta}{1-\delta e_1}\left(e_2 z_2^2 - e_3 z_2^3\right)$	$1 - z_1^2 - z_3^2$	$\frac{\delta}{1-\delta e_3}\left(e_1 z_2^1 - e_2 z_2^2\right)$
z_3	$\frac{\delta}{1-\delta e_1}\left(e_2 z_3^2 - e_3 z_3^3\right)$	$\frac{\delta}{1-\delta e_2}\left(e_1 z_3^1 - e_3 z_3^3\right)$	$1 - z_1^3 - z_2^3$

Table 4: Generic shares with three parties

By solving the system of equations, we can (for each of the eight scenarios) compute the continuation value for each party. In particular, for the case {1,3}, {2,3} and {3,2}, results are summarised in table 5.

	Formateur		
Shares	1	2	3
z_1	$\frac{(1-\delta)(1-\delta e_3)}{1-\delta+\delta^2 e_1 e_3}$	0	0
z_2	0	$\frac{(1-\delta)(1-\delta e_3)}{1-\delta+\delta^2 e_1 e_3}$	$\frac{(1-\delta)\delta e_2}{1-\delta+\delta^2 e_1 e_3}$
z_3	$\frac{(1-\delta e_2-\delta e_3)\delta e_3}{1-\delta+\delta^2 e_1 e_3}$	$\frac{(1-\delta e_2-\delta e_3)\delta e_3}{1-\delta+\delta^2 e_1 e_3}$	$\frac{(1-\delta)(1-\delta e_2)+\delta^2 e_1 e_3}{1-\delta+\delta^2 e_1 e_3}$

Table 5: Shares at equilibrium

The continuation value depends on the coalition and on the identity of the formateur. To have a stable equilibrium, each party always chooses to form the same coalition when it is the formateur. For stability, its choice must be, at every period of time, the best response to others' player behaviour and the strategy has to be always the same. Committing to a given strategy allows parties to modify their continuation value when they are not formateur.

Within the eight scenarios, we look for Nash simultaneous stable subgame perfect equilibria in pure strategies (SSPPS). Each player has two possible actions (consisting in forming a coalition with either of the remaining parties). Comparing expected payoffs of each party in each situation (through a reduced form game matrix of payoff), we notice that only scenario $(\{1,3\},\{2,3\},\{3,2\})$ is SSPPS. If

[40]Note that, according to which coalition is formed, some of the cells in the table take the value zero.

for example party 2 always form a coalition {2,3} and party 3 a coalition {3,2}, then for party 1 it is a dominating strategy to form a coalition {1,3}.

To check that this scenario is really an equilibrium, take the generic recognition probabilities (a, b, c). By definition of continuation value, $v_j^* = az_j^1 + bz_j^2 + cz_j^3$ thus:

$$\begin{cases} v_1 = \frac{a(1-\delta)(1-\delta c)}{1-\delta+\delta^2 ac} + 0 + 0 \\ v_2 = 0 + b\frac{(1-\delta)(1-\delta c)}{1-\delta+\delta^2 ac} + c\frac{(1-\delta)\delta b}{1-\delta+\delta^2 ac} = \frac{b(1-\delta)}{1-\delta+\delta^2 ac} \\ v_3 = (a+b)\,\frac{(1-\delta b-\delta c)\delta c}{1-\delta+\delta^2 ac} + c\frac{(1-\delta)(1-\delta b)+\delta^2 ac}{1-\delta+\delta^2 ac} = \frac{(1-\delta c-\delta b)c}{1-\delta+\delta^2 ac} \end{cases}$$

and thus $v = \left(\frac{a(1-\delta)(1-\delta c)}{1-\delta+\delta^2 ac}; \frac{b(1-\delta)}{1-\delta+\delta^2 ac}; \frac{(1-\delta c-\delta b)c}{1-\delta+\delta^2 ac}\right)$.

For $a = e_1$, $b = e_2$, $c = e_3$, and knowing that $z_j^i = \delta v_j$ for $i \neq j$, we are back to results in table 5.

We check now that no player wants to deviate: refer to an equilibrium E via the corresponding formed coalition when a party is formateur ,call E^* the above proposed equilibrium (that is $(\{1,3\},\{2,3\},\{3,2\})$). Define then E^i the alternative candidate equilibrium if party i deviates.

Since we look for stationary pure strategy equilibria, to show that no player wants to deviate, I show that a) $E^* \succ_1 E^1 = (\{1,2\},\{2,3\},\{3,2\})$, b) $E^* \succ_2 E^2 = (\{1,3\},\{2,1\},\{3,2\})$ and c) $E^* \succ_3 E^3 = (\{1,3\},\{2,3\},\{3,1\})$.

a) $E^* \succ_1 E^1$ iff $(v_3 \mid E^*) < (v_2 \mid E^1)$, which means $\frac{(1-\delta e_2-\delta e_3)e_3}{1-\delta+\delta^2 e_1 e_3} < \frac{(1-\delta e_2-\delta e_3)e_2}{1-\delta+\delta^2 e_1 e_2}$, thus $e_3\,(1-\delta+\delta^2 e_1 e_2) < e_2\,(1-\delta+\delta^2 e_1 e_3)$. Since $e_3 < e_2$, it is clear that $e_3\,(1-\delta) < e_2\,(1-\delta)$.

b) $E^* \succ_2 E^2$ iff $(v_3 \mid E^*) < (v_1 \mid E^2)$, which means $\frac{(1-\delta e_2-\delta e_3)e_3}{1-\delta+\delta^2 e_1 e_3} < \frac{(1-\delta e_1-\delta e_3)e_1}{1-\delta+\delta^2 e_2 e_3}$. From $0.5 > e_1 > e_2 > e_3$, it is a matter of simple algebra to show that the left hand side is always smaller than the right hand side.

c) $E^* \succ_3 E^3$ iff $(v_2 \mid E^*) < (v_1 \mid E^3)$, which means $\frac{(1-\delta)e_2}{1-\delta+\delta^2 e_1 e_3} < \frac{(1-\delta)e_1}{1-\delta+\delta^2 e_2 e_3}$. Result follows directly.

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