Asymptotic Expansions of the QMLEs in the EGARCH(1,1) Model*

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Abstract

In this paper we develop the Edgeworth expansions of the Quasi Maximum Likelihood Estimators (QMLEs) of the EGARCH$(1,1)$ parameters and we derive their asymptotic properties, in terms of bias and mean squared error. The notions of geometric ergodicity and stationarity are also discussed in shedding light on the asymptotic theory of the EGARCH models. We also examine the effect on the QMLEs of the variance parameters by including an intercept in the mean equation. We check our theoretical results by simulations. In doing this, we employ either analytic or numeric derivatives and the convergence properties of the methods will be also discussed.

Keywords: Edgeworth expansion; quasi-maximum likelihood; EGARCH model; bias approximations; asymptotic properties.

JEL classification numbers: C13; C22

1 Introduction

The last years there has been a substantial interest in approximating the exact distributions of econometric estimators in time series models and deriving their asymptotic properties. Although there is an important and growing literature that deals with the asymptotics of the Generalized Autoregressive Conditional Heteroskedastic (GARCH) models, either in terms of consistency and asymptotic normality of the estimators or in terms of the finite-sample theory, the asymptotic properties of the estimators in the Exponential GARCH (EGARCH) process of Nelson [22] have not been fully explored.

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Comparing to the GARCH process, the advantages of the EGARCH model are well-known, with the main one being the fact that the model captures the negative dynamic asymmetries noticed in many financial series, i.e. the so-called leverage effects.

The asymptotic aspects of the conditionally heteroskedastic models have been discussed under many different considerations, in order to analyze the statistical properties of these estimators. Since the important work of Engle [11] and that of Bollerslev [2], who introduced the Autoregressive Conditional Heteroskedasticity (ARCH) and Generalized ARCH models, respectively, a huge amount of literature on the asymptotics has appeared in short time. Weiss [28] proved Consistency and Asymptotic Normality (CAN) of the maximum likelihood estimators in ARCH models, assuming normal distribution of the errors and imposing a rather restrictive condition that the data have bounded fourth moments, excluding in that way from the proof many other interesting conditionally heteroskedastic models. Quite parallel, Lee and Hansen [16] and Lumsdaine [18] relaxed the condition which Weiss imposed and they looked at the consequences of the possible failure of the normality assumption on the errors, providing conditions under which CAN exist in the GARCH$(1,1)$ specification (for multivariate frameworks see e.g. Jeantheau [15]; Comte and Lieberman [6]).

The problem of proving stationarity in the GARCH context has not been solved until Nelson [21] provided conditions for the existence and uniqueness of a stationary solution in the GARCH$(1,1)$ case. Mixing and moment properties have been investigated for various GARCH and stochastic volatility models, see Carrasco and Chen [5]. The small sample properties of the estimators in the first order GARCH model are investigated through an asymptotic expansion of the Edgeworth type, as Linton [17] developed[1]. Nowadays, many researchers work on the asymptotic behaviour of these estimators, with unceasing interest.

Until the influential work of Nelson [22], the conditional heteroskedastic models that had been developed could not explain the asymmetry effects, indicating that alternative models might be suitable for financial applications. Turning our attention to asymmetric GARCH models, and more specifically to the EGARCH model which has become a popular model in applied work, very little is known about its statistical properties. Although we are endowed with the moment structure investigated by He, Terasvirta and Malmsten [13], the limiting properties of the maximum likelihood estimators do not exist in the literature. The interest in consistency and asymptotic normality results of EGARCH has been growing and the problem of the theoretical properties not yet been explored await for an answer; see, for example, Straumann and Mikosch [26].

In this paper we develop the Edgeworth expansions of the Quasi Maximum Likelihood Estimators (QMLEs) of the EGARCH$(1,1)$ parameters and we derive their asymptotic properties, in terms of bias and mean squared error. The notions of geometric ergodicity and stationarity are also discussed in shedding light on the asymptotic theory of the EGARCH models. We also examine the effect on the QMLEs of the variance parameters by including an intercept in the mean equation. We check our theoretical results by simulations. In doing this, we employ either analytic or numeric derivatives and the convergence properties of the methods will be also discussed.

1.1 Literature Review on the EGARCH model

The literature on GARCH models is really vast, so that readers are refered for example to the survey papers of Bera and Higgins [1] and Bollerslev, Chou and Kroner [3], as well as Bollerslev, Engle and Nelson [4]. These papers review the properties and the estimation techniques, as well as the statistical tests that have been developed and they comprise the origin that all researchers in this field should begin with. Here, we review some papers that mostly deal with the features and the statistical properties of the EGARCH model.

In his seminal paper, Nelson [22] pointed out that the simple structure of the GARCH and related processes, i.e. symmetric non-linear models, is not the most important criterion in modelling the financial time series. He underlined that there are some unexceptionable drawbacks of the GARCH theory indicating that a new approach is necessary to be established. The Exponential GARCH model of Nelson meet these limitations, which can be summarized thereupon. First, volatility tends to rise in response to "bad news" and fall in response to "good news", so that a sign effect should be introduced in the model. Second, the strict assumption of nonnegative coefficients is not imposed as the logarithm of the conditional variance ensures that there is always positivity. Third, interpretation of the persistence of shocks to conditional variance is possible through the EGARCH context. Nelson also derived a necessary and sufficient condition for strict stationarity of the EGARCH process, when $\ln h_t^2$ has an infinite moving average representation (see Theorem 2.1, p. 351).

Moreover, in the paper of He, Terasvirta and Malmsten [13] we are given the moment structure of the standard EGARCH$(1,1)$ model and without assuming normality of the errors. The existence of unconditional moments and the kurtosis, as well as the properties of the autocorrelation function of positive powers of absolute observations are examined. The results presented there are important for comparisons with the GARCH model. For an analysis which assumes normality and presents an alternative technique for the expression of the autocorrelation function of squared observations, the reader is refered to the paper of Demos [10].

The finite sample properties of the maximum likelihood and quasi-maximum likelihood estimators of the EGARCH$(1,1)$ process using Monte Carlo methods have been examined in the paper of Deb [9]. He used, however, response surface methodology in order to examine the finite sample bias and other properties in interest.

Recently, Dahl and Iglesias [8] analysed the limiting properties, in terms of consistency and asymptotic normality, of the estimated parameters in a related but in many aspects different to the traditional EGARCH model of Nelson, [22]. More specifically, they allow for a variant specification of the conditional variance in which the standardised residuals (z_{t-1}), which are introduced in the next section, are replaced by the observed process and as a consequence the recursive nature of z_{t-1} in the conditional variance function no longer exists. Automatically, this alteration makes the whole proofs easier, but the investigation of the asymptotic properties of Nelson's model still remains unsolved.

Another related paper is that of Perez and Zaffaroni [23] who derived and compared the finite sample properties of maximum likelihood and Whittle estimators in EGARCH models. This paper considers also the Fractionally Integrated EGARCH (FIEGARCH) model to represent the long-memory behaviour observed in the correlations of squared returns. The results of the paper are illustrated by a Monte Carlo experiment. The outcome of such an analysis is that if someone performs Whittle estimation, its performance is comparable to maximum likelihood, especially when considering the bias of the estimates, while the maximum likelihood is often superior in terms of mean squared error.

To the best of our knowledge, in terms of consistency and asymptotic normality, the theoretical properties of the QMLEs in the EGARCH model have not been studied in the literature. Straumann and Mikosch [26] established the asymptotic properties of this model but of a lower order, with stochastic recurrence equations methods. They encountered difficulties of establishing invertibility of the model, which is critical in order to make sure that the likelihood function is well-behaved asymptotically.

The organisation of the paper is as follows: Section 2 presents the model and estimators. Section 3 deals with the main results of our analysis. First, analytic derivatives and their expected values are presented. Second, conditions for stationarity of the log-variance derivatives are investigated. In the sequel, the Edgeworth expansions of the Quasi Maximum Likelihood Estimators are derived and theoretical bias approximations of the estimators are calculated. Finally, Section 4 concludes. All proofs, rather lengthy, are collected in the Appendix at the end. Let us now turn our attention to the definition of the EGARCH$(1,1)$ model and its statistical properties.

2 The Model and Estimators

Let us consider the following model, where the observed data $\{y_t\}_{t=1}^{T}$ are generated by the EGARCH$(1,1)$ process, see Nelson [22], in which the conditional variance, h_t, depends on both the size and the sign of the lagged residual:

$$\begin{aligned} y_t &= \mu + u_t, \qquad t = 1, ...T, \quad where & (1) \\ u_t &= z_t \sqrt{h_t}, \\ u_t | \mathcal{A}_{t-1} &\sim N(0, h_t), \quad z_t \sim iid(0,1) \\ \ln(h_t) &= \alpha + \theta z_{t-1} + \gamma g(z_{t-1}) + \beta \ln(h_{t-1}), \qquad where & (2) \\ g(z_t) &= |z_t| - E|z_t| \end{aligned}$$

$\{u_t\}$ is a real-valued discrete time stochastic process (the error process) and h_t is a positive with probability one $\mathcal{A}_{t-1}$-measurable function (the conditional variance), where $\mathcal{A}_{t-1}$ is the sigma-algebra generated by the past values of z_t, i.e. $\{z_{t-1}, z_{t-2}, z_{t-3}, ...\}$. The function $g(z_t)$ is a well-defined function of z_t. The process h_t is not observed and thus is constructed via recursion using the estimating values of the parameters and a proper initial value for the conditional variance. The only distributional assumption made about the innovations $z_t's$ is that they are independently and identically distributed (iid) with zero mean and unit variance. We do not impose any symmetric distributional property, however the proofs automatically become very tedious, but also very challenging. The conditional variance is constrained to be non-negative by the assumption that the logarithm of h_t is a function of past $z_t's$. Comparing to relative research, the Nelson's paper was the first which models the conditional variance as a function of variables which are not solely squares of the observations.

It is useful to rewrite our model in the multiplicative form; from definition (2) it follows that

$$\begin{aligned} h_t &= \exp\{\alpha + \theta z_{t-1} + \gamma g(z_{t-1}) + \beta \ln(h_{t-1})\} = \\ &= \exp\{\alpha\} \exp\{\theta z_{t-1}\} \exp\{\gamma g(z_{t-1})\} h_{t-1}^{\beta}. & (3) \end{aligned}$$

The equations of the EGARCH process create a complicated probabilistic structure which is not easily understood. Note from eq. (2) that $\ln(h_t)$ constitutes a causal AR(1) process with mean $\alpha/(1-\beta)$ and error sequence $[\theta z_{t-1} + \gamma(|z_{t-1}| - E|z_{t-1}|)]$. The unique stationary solution to (2), provided that $|\beta| < 1$, is given by its almost sure (a.s.) representation:

$$\begin{aligned} \ln(h_t) &= \alpha(1-\beta)^{-1} + \sum_{k=0}^{\infty} \beta^k (\theta z_{t-1-k} + \gamma g(z_{t-1-k})) \Rightarrow \\ \ln(h_t) &\geq \alpha(1-\beta)^{-1} \qquad a.s. \end{aligned}$$

The conditional variance responds asymmetrically to rises and falls in stock price, which is believed to be important for example in modelling the behaviour of stock returns. It is an important stylized fact for many assets. The coefficients $(\theta + \gamma)$ and

$(\theta - \gamma)$ (if $z_t \geq 0$ and $z_t < 0$, respectively) show the asymmetry in response to positive and negative y_t. The parameter θ is referred to as the leverage parameter, which shows the effect of the sign of y_t. The term $\gamma \left[|z_t| - E|z_t| \right]$ represents a magnitude effect. Formulae for the higher order moments of u_t are given in Nelson [22]. The parameter α can be made a function of time (α_t) to accommodate the effect of any non-trading periods of forecastable effects.

The unconditional mean and variance of y_t is:

$$E\left(y_t\right) = \mu,$$

and

$$Var\left(y_t\right) = \exp\left(\frac{\alpha}{1-\beta}\right) \prod_{i=0}^{\infty} E\left[\exp\left[\beta^i\left(\theta z_0 + \gamma g\left(z_0\right)\right)\right]\right],$$

which, under normality of the errors, becomes the following result:

$$Var\left(y_t\right) = \exp\left(\frac{\alpha - \gamma\sqrt{\frac{2}{\pi}}}{1-\beta}\right) \prod_{i=0}^{\infty} \left[\exp\left(\frac{\beta^{2i}\left(\gamma^*\right)^2}{2}\right)\Phi\left(\beta^i\gamma^*\right) + \exp\left(\frac{\beta^{2i}\delta^2}{2}\right)\Phi\left(\beta^i\delta\right)\right],$$

where $\gamma^* = \gamma + \theta$, $\delta = \gamma - \theta$ and $\Phi\left(k\right)$ is the value of the cumulative standard Normal evaluated at k, i.e. $\Phi\left(k\right) = \int_{-\infty}^{k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$.

Proof. The proof of the unconditional variance is given in the Appendix. ■

The last expressions above may be difficult to compute. We have to approximate the infinite products by a product with a finite number of terms. Such expressions will arise in many different points below in our analysis.

To estimate the parameters of the model in eq. (1) and eq. (2), we employ the quasi-maximum likelihood estimation. Maximum likelihood is the procedure which is most often used in estimating the parameters in time series models, but for most applications it is very difficult to justify the conditional normality assumption. Therefore, the log-likelihood function may be misspecified. However, we can still obtain estimates by maximizing a Gaussian quasi-log-likelihood function and under the auxiliary assumption of an iid distribution for the standardized innovations $z_t's$. The estimators which are derived by this maximization problem are the so-called Quasi Maximum Likelihood Estimators (QMLEs). The fact that the we maximize a quasi-log-likelihood is justified by the evidence that distributions of asset returns are often thick tailed and as a consequence the normality assumption is violated and hence the likelihood is not well specified.

An important and really interesting feature of our model is that the assumption of the block diagonality of the information matrix no longer holds. This is also the case for the ARCH-M model and the asymmetric model of the Augmented ARCH (see Bera and Higgins [1], p. 349; also Bollerslev, Engle and Nelson [4], p. 2981). This implies that the off-diagonal blocks involving partial derivatives with respect to both mean and variance parameters are not null matrices, while this is the case in other GARCH-type models. Below we present analytic proofs of this argument in the context of the EGARCH$(1,1)$ model and these results disaccord with Malmsten [19], even if the distribution of the innovations is symmetric, which implies that $Ez^3 = 0$.

Nelson [22] proposes to estimate the EGARCH model by maximum likelihood assuming that the innovations have a Generalized Error Distribution (GED). The appeal of this distribution comes from the fact that it nests the normal distribution as a special case, and can also display both thinner and thicker tails than the normal, depending on the value achieved by the tail thickness parameter[2].

In the EGARCH$(1,1)$ model, there is no explicit expression of the probability density of the vector $(y_1, ..., y_T)'$ since the distribution of $(h_1, ..., h_T)'$ is not known. To overcome this difficulty, we consider an approximate conditional log-likelihood instead. Some assumptions are also required for the initial values of the conditional variance h_t, which should be drawn from the stationary distribution, and the squared standardized residuals z_t^2. Assuming that $z_0 = 0$ and $\ln(h_0) = \frac{\alpha}{1-\beta}$, we obtain a good approximation to the conditional Gaussian log-likelihood, as follows:

$$\begin{aligned}\ell\left(\mu,\alpha,\theta,\beta,\gamma\right| z_0,h_0) &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \sum_{t=1}^{T}\frac{(y_t-\mu)^2}{2h_t} = \\ &= -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T}\ln(h_t) - \frac{1}{2}\sum_{t=1}^{T}z_t^2. \qquad (4)\end{aligned}$$

Notice that h_t and z_t are both functions of $\boldsymbol{\omega}$ and μ, where $\boldsymbol{\omega} = (\alpha,\theta,\beta,\gamma)^{/}$, i.e. the vector of unknown log-variance parameters, so that both are functions of $\boldsymbol{\varphi} = \left(\boldsymbol{\omega}^{/},\mu\right)^{/}$, which represents the vector of all unknown parameters. The first order conditions are recursive and consequently do not have explicit solutions.

The likelihood function is derived as though the errors are conditionally normal and is still maximized at the true parameters. Having specified the log-likelihood function, the quasi maximum likelihood estimator is then defined as

$$\widehat{\theta_T} = \arg\max_{\theta\in\Theta}\frac{1}{T}\sum_{t=1}^{T}\ell(\theta). \qquad (5)$$

Because the likelihood function is misspecified, the form of the covariance matrix of the QMLEs is more complex than the usual inverse of the information matrix, see White [29] and [30]. More specifically, if we assume that the random functions $h_t(\boldsymbol{\varphi})$

are almost surely twice continuously differentiable then the following matrices $\mathbf{G}_0$ and $\mathbf{F}_0$ are well-defined:

$$\begin{aligned}\mathbf{F}_0 &= -\frac{2}{E\left(z_0^4-1\right)}\mathbf{G}_0,\\ \mathbf{G}_0 &= \frac{E\left(z_0^4-1\right)}{4}E\left(\frac{1}{h_0^2}\left(\frac{\partial h_0\left(\boldsymbol{\varphi}_0\right)}{\partial\boldsymbol{\varphi}}\right)'\left(\frac{\partial h_0\left(\boldsymbol{\varphi}_0\right)}{\partial\boldsymbol{\varphi}}\right)\right).\end{aligned}$$

The asymptotic covariance matrix is given by $\mathbf{F}_0^{-1}\mathbf{G}_0\mathbf{F}_0^{-1}$. If z_0 is actually standard Gaussian, then $\mathbf{G}_0 = -\mathbf{F}_0$ and thus the covariance matrix asymptotically is given by the well-known result of the inverse of the information matrix. There are strongly consistent estimators of the matrices $\mathbf{G}_0$ and $\mathbf{F}_0$; indeed:

$$\widehat{\mathbf{A}_n} = \frac{1}{n}\left(\mathcal{L}_i\right)'\left(\mathcal{L}_i\right) \overset{a.s.}{\rightarrow} \mathbf{G}_0, \qquad \widehat{\mathbf{B}_n} = \frac{1}{n}\mathcal{L}_{ij} \overset{a.s.}{\rightarrow} \mathbf{F}_0,$$

where $\mathcal{L}_i = \frac{\partial\ell(\boldsymbol{\varphi}_0)}{\partial\boldsymbol{\varphi}}$ and $\mathcal{L}_{ij} = \frac{\partial^2\ell(\boldsymbol{\varphi}_0)}{\partial\boldsymbol{\varphi}\partial\boldsymbol{\varphi}'}$, for $i, j \in \{\mu, \alpha, \theta, \beta, \gamma\}$.

In summary, the steps in the estimation method are the following. If the type of the distribution of z_t is not specified, it is impossible to determine a likelihood function. A common approach in such a situation is to suppose that $z_t \sim iidN\left(0,1\right)$. Now it is possible to determine a so-called Gaussian quasi log-likelihood and solving a maximization problem to derive the quasi maximum likelihood estimator. Instead of using the exact log-likelihood, we concentrate on finding an approximation to a conditional likelihood. We then make use of proper initial values in order to obtain a good approximation to the conditional likelihood. In that way, we derive an estimator which is asymptotically equivalent to the QMLE that we obtained as the solution of the maximization problem.

Let us proceed with the main results of our analysis, beginning with the analytic derivatives of the log-likelihood function and their expected values.

3 The Main Results

3.1 Analytic derivatives and their expected values

In this section we present the derivatives of the log-likelihood function and their expected values, which are needed in the sequel to calculate the cumulants of the Edgeworth distribution and to evaluate the asymptotic bias of the QMLEs. It is of great importance to mention that there are no such analytic results in the literature, and it is especially this feature that makes this analysis to differ from the previous one, that of Linton [17], who studied the case of the GARCH$(1,1)$ model.

Before presenting our first results, it is worthwhile to discuss the usefulness of the analytic derivatives and their expected values, in terms of the asymptotic theory. The limit behaviour of the sequences of functions of first and second order derivatives of the log-likelihood function are essential to be studied for establishing the asymptotic normality of the QMLEs. By means of a Taylor expansion of the score function of the log-likelihood, the asymptotic normality is established. Consequently, one has to study the differentiability properties of the conditional variance, which is a priliminary but crucial step in large-sample theory. Moreover, the expected values of the derivatives of the log-likelihood are needed in order to calculate the cumulants of the approximate distribution and these results are then used to calculate the bias expressions of the estimators. Let us first proceed with the derivatives of the log-likelihood function and their analytic representation.

Following the notation employed in Linton (1997), i.e. $h_{t;\circ} = \frac{\partial \ln(h_t)}{\partial \circ}$, the derivatives of the log-likelihood function with respect to all the parameters are:

$$
\begin{aligned}
\mathcal{L}_\mu &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu} + \sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}, \\
\mathcal{L}_{\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu} - \sum_{t=1}^{T}\left(\frac{1}{h_t} + 2\frac{z_t}{\sqrt{h_t}}h_{t;\mu} + \frac{1}{2}z_t^2 h_{t;\mu}^2\right), \\
\mathcal{L}_{\mu\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu,\mu} + 3\sum_{t=1}^{T}\frac{1}{h_t}h_{t;\mu} \\
&\quad -3\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;\mu,\mu}-h_{t;\mu}^2\right) - \frac{1}{2}\sum_{t=1}^{T}z_t^2\left(3h_{t;\mu}h_{t;\mu,\mu}-h_{t;\mu}^3\right)
\end{aligned}
$$

while for $i, j, k \in \{\alpha, \theta, \gamma, \beta\}$ the derivatives are:

$$
\begin{aligned}
\mathcal{L}_i &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}, \\
\mathcal{L}_{ij} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j} - \frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;i}^2, \\
\mathcal{L}_{ijk} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j,k} - \frac{1}{2}\sum_{t=1}^{T}z_t^2\left(3h_{t;i}h_{t;j,k}-h_{t;i}^3\right).
\end{aligned}
$$

The cross derivatives are given by the following expressions:

$$
\begin{aligned}
\mathcal{L}_{i\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;i}h_{t;\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i},\\
\mathcal{L}_{i\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,\mu,\mu}-2\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;i,\mu}-h_{t;i}h_{t;\mu}\right)\\
&\quad -\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(2h_{t;\mu}h_{t;i,\mu}-h_{t;i}h_{t;\mu}^2+h_{t;i}h_{t;\mu,\mu}\right)+\sum_{t=1}^{T}\frac{1}{h_t}h_{t;i},\\
\mathcal{L}_{ij\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j,\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\left(h_{t;i,j}-h_{t;i}h_{t;j}\right)\\
&\quad -\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;j}h_{t;i,\mu}-h_{t;j}h_{t;i}h_{t;\mu}+h_{t;i,j}h_{t;\mu}+h_{t;i}h_{t;j,\mu}\right).
\end{aligned}
$$

Note that the log-likelihood derivatives are expressions of the log-variance derivatives, $h_{t;\circ}$, where the latter are given in the Appendix. The expected values of the log-likelihood derivatives are also given in the Appendix.

The cross-products of the log-likelihood derivatives are:
For $i,j\in\{\alpha,\theta,\gamma,\beta\}$,

$$
\begin{aligned}
\mathcal{L}_i\mathcal{L}_{ij} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;i}^2\right),\\
\mathcal{L}_i\mathcal{L}_{j\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;j,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2 h_{t;j}h_{t;\mu}-\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i}\right),\\
\mathcal{L}_i\mathcal{L}_{\mu\mu} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i}\left[\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}-\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2h_{t;\mu}^2\right)\right],\\
\mathcal{L}_\mu\mathcal{L}_{ij} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;i,j}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;i}^2\right),\\
\mathcal{L}_\mu\mathcal{L}_{j\mu} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left(\begin{array}{c}\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;j,\mu}-\frac{1}{2}\sum_{t=1}^{T}z_t^2h_{t;j}h_{t;\mu}\\ -\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}h_{t;i}\end{array}\right),\\
\mathcal{L}_\mu\mathcal{L}_{\mu\mu} &= \left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu}+\sum_{t=1}^{T}\frac{z_t}{\sqrt{h_t}}\right)\left[\begin{array}{c}\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\mu,\mu}\\ -\sum_{t=1}^{T}\left(\frac{1}{h_t}+2\frac{z_t}{\sqrt{h_t}}h_{t;\mu}+\frac{1}{2}z_t^2h_{t;\mu}^2\right)\end{array}\right].
\end{aligned}
$$

The expectations of the cross-products are given in the Appendix.

Let us turn our attention to the conditions for stationarity of the log-variance derivatives.

3.2 Conditions for stationarity of the log-variance derivatives

In this section we investigate under which conditions there is a stationary solution to the log-variance derivatives, needed for the existence and the evaluation of the log-likelihood derivatives, and hence in order to calculate the bias expressions of the QMLEs. The existence, stationarity and ergodicity of the second order derivatives of the conditional variance are necessary so that the Taylor expansion of the first order derivatives of the log-likelihood is validated.

Let the following:

$$\begin{aligned} h_{t;\alpha}h_{t;\alpha\alpha} &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h^2_{t-1;\alpha}+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h^3_{t-1;\alpha} \\ &+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha} \\ &+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 h_{t-1;\alpha}h_{t-1;\alpha,\alpha}. \end{aligned} \tag{6}$$

In order to calculate the expected value of the above expression, we first assume that $E\left(h^2_{t;\alpha}\right), E\left(h^3_{t;\alpha}\right)$ and $E\left(h_{t;\alpha,\alpha}\right)$ exist. Next, define:

$$\begin{aligned} A\left(z_{t-1}\right) &= \frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)h^2_{t-1;\alpha} \\ &+\frac{1}{4}\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h^3_{t-1;\alpha} \\ &+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha,\alpha}, \end{aligned}$$

and

$$B^2\left(z_{t-1}\right)=\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2.$$

Then,

$$\begin{aligned} h_{t;\alpha}h_{t;\alpha\alpha} &= A\left(z_{t-1}\right)+B^2\left(z_{t-1}\right)h_{t-1;\alpha}h_{t-1;\alpha,\alpha}= \\ &= A\left(z_{t-1}\right)+\sum_{k=1}^{\infty}\prod_{i=0}^{k-1}B^2\left(z_{t-1-i}\right)A\left(z_{t-1-k}\right). \end{aligned}$$

The infinite sum converges almost surely. To see this, let:

$$S_n = A\left(z_{t-1}\right) + \sum_{k=1}^{n} \prod_{i=0}^{k-1} B^2\left(z_{t-1-i}\right) A\left(z_{t-1-k}\right).$$

Then we have:

$$\begin{aligned} E\left(S_n\right) &= E\left[A\left(z_{t-1}\right)\right] + \sum_{k=1}^{n} E\left[\prod_{i=0}^{k-1} B^2\left(z_{t-1-i}\right)\right] E\left[A\left(z_{t-1-k}\right)\right] = \\ &= E\left[A\left(z_{t-1}\right)\right]\left[\sum_{k=0}^{n}\left\{E\left[B^2\left(z_{t-1-i}\right)\right]\right\}^k\right]. \end{aligned}$$

Thus, $E\left(\lim_{n\to\infty} S_n\right) = E\left[A\left(z_{t-1}\right)\right]\left\{1 - E\left[B^2\left(z_{t-1-i}\right)\right]\right\}^{-1} < \infty$, providing that $E\left[A\left(z_{t-1}\right)\right] < \infty$. In order to ensure the existence of a stationary solution to the eq. (6), we should impose the condition that

$$\left|\left[B^2\left(z_{t-1-i}\right)\right]\right| < 1.$$

In a similar manner, the rest stationarity conditions of all log-variance derivatives and the products of them follow. Let us now proceed with the asymptotic expansions of the QMLEs.

3.3 Asymptotic Expansions of the QMLEs

There are relatively few papers that theoretically investigate the small sample properties of estimators in non-linear time series models due to the difficulty of the analysis and the awkward expressions that one can end up with (e.g. Iglesias and Phillips [14]). In particular, analytical work on bias and mean squared error of the estimators in the general context of GARCH models has not received much attention in the literature, until Linton [17] calculated the formal Edgeworth distribution function for the GARCH$(1,1)$ process.

Due to the difficult expressions for the bias of the estimators, Linton used numerical integration in order to evaluate their magnitude. We make one step further and present analytic results of the expected values of the log-likelihood derivatives needed in the sequel in order to calculate the bias approximations. These expressions are presented in such a form so that one may choose a specific distribution for the errors and end up with the desired results. These theoretical outcomes can then be checked through simulations in terms of accuracy and be compared with numeric derivatives.

The Edgeworth expansions have a long history in econometrics[3]. Taniguchi, [27], developed the expansions for many time series models. The Edgeworth expansion in the likelihood context is derived through a specific method, which expands the score functions of the log-likelihood in a power series in the likelihood derivatives about the true parameter value and then inverts this expansion to yield an expansion for the standardized estimator. Let $\widehat{\theta}$ be an estimator of θ and

$$\overline{\theta} = \sqrt{n}\left(\widehat{\theta} - \theta\right)$$

The Edgeworth expansion of $\overline{\theta}$, with an error of $O\left(T^{-1}\right)$, is given by

$$P\left(\overline{\theta} \leq m\right) = \Phi\left(\frac{m}{\omega}\right) - \phi\left(\frac{m}{\omega}\right)\left[\psi_0 + \psi_1\left(\frac{m}{\omega}\right) + \psi_2\left(\frac{m}{\omega}\right)^2\right],$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right)$ and $\Phi(\alpha) = \int_{-\infty}^{\alpha}\phi(z)\,dz$.

That is, as we express $\sqrt{T}\left(\widehat{\theta} - \theta\right)$ as a function of the first and second order derivatives of the log-likelihood, standardised appropriately and evaluated at the true parameter values, we are then able to calculate the approximate Edgeworth distribution.

Let $\overline{\varphi} = \left(\overline{\theta}_1, \overline{\theta}_2, \overline{\theta}_3, \overline{\theta}_4, \overline{\theta}_5\right)^{/} =$ $\left(\sqrt{T}\left(\widehat{\mu} - \mu\right), \sqrt{T}\left(\widehat{\alpha} - \alpha\right), \sqrt{T}\left(\widehat{\theta} - \theta\right), \sqrt{T}\left(\widehat{\gamma} - \gamma\right), \sqrt{T}\left(\widehat{\beta} - \beta\right),\right)^{/}$. Consider the Taylor expansion for $\frac{1}{\sqrt{T}}\frac{\partial\ell(\widehat{\varphi})}{\partial\varphi}$, where $\widehat{\varphi} = \left(\widehat{\mu}, \widehat{\alpha}, \widehat{\theta}, \widehat{\gamma}, \widehat{\beta}\right)^{/}$ around the true value $\varphi = (\mu, \alpha, \theta, \gamma, \beta)^{/}$ as:

$$0 = \frac{1}{\sqrt{T}}\frac{\partial\ell}{\partial\theta_j} + \frac{1}{T}\sum_{k=1}^{5}\frac{\partial^2\ell(\varphi)}{\partial\theta_j\partial\theta_k}\overline{\theta}_k + \frac{1}{2T^{3/2}}\sum_{k,l=1}^{5}\frac{\partial^3\ell(\varphi^*)}{\partial\theta_j\partial\theta_l\partial\theta_k}\overline{\theta}_k\overline{\theta}_l,$$

where $j = 1, ..., 5$ and $\frac{1}{\sqrt{T}}\frac{\partial\ell}{\partial\theta_j}$, $\frac{\partial^2\ell}{\partial\theta_j\partial\theta_l}$ are evaluated at the true values and $\frac{\partial^3\ell}{\partial\theta_j\partial\theta_l\partial\theta_k}$ is evaluated at an intermediate point, say φ^*, between the estimates and the true values. The above expression can be written as

$$\begin{aligned} 0 &= \frac{1}{\sqrt{T}}\frac{\partial\ell}{\partial\theta_j} + \sum_{i=1}^{5}\left(A_{ji} + \frac{w_{ji}}{\sqrt{T}}\right)\overline{\theta}_i + \frac{1}{2\sqrt{T}}\sum_{k,i=1}^{5}K_{jik}\overline{\theta}_i\overline{\theta}_k + O_p\left(T^{-1}\right) \\ &\equiv f_j\left(\overline{\varphi}, v\right) + O_p\left(T^{-1}\right), \quad j = 1, ..., 5 \end{aligned}$$

where $A_{ij} = \frac{1}{T}E\left(\frac{\partial^2\ell(\varphi)}{\partial\theta_j\partial\theta_i}\right)$, $K_{jik} = \frac{1}{T}E\left(\frac{\partial^3\ell(\varphi)}{\partial\theta_j\partial\theta_i\partial\theta_k}\right)$,

$$w_{ij} = \frac{1}{\sqrt{T}}\left(\frac{\partial^2\ell(\varphi)}{\partial\theta_j\partial\theta_i} - TA_{ij}\right),$$

for $i, j, k = 1, ..., 5$. Let us define a vector v containing the non-zero elements of $\frac{1}{\sqrt{T}}\frac{\partial \ell}{\partial \theta_i}$ and w_{ij}, for $i, j = 1, ..., 5$. Solving for $\overline{\theta}_j$, and $j = 1, ..., 5$, as continuously differentiable functions of v, gives:

$$\begin{aligned} \overline{\theta}_j\left(v\right) &= \sum_{a=1}^{q} \frac{\partial \overline{\theta}_j\left(0\right)}{\partial v_a} v_a + \frac{1}{2} \sum_{a,b=1}^{q} \frac{\partial^2 \overline{\theta}_j\left(0\right)}{\partial v_a \partial v_b} v_a v_b + O_p\left(T^{-1}\right) \\ &\equiv \sum_{a=1}^{q} e_a^{(j)} v_a + \frac{1}{2\sqrt{T}} \sum_{a,b=1}^{q} e_{ab}^{(j)} v_a v_b + O_p\left(T^{-1}\right) \end{aligned}$$

where $e_a^{(j)} = \frac{\partial \overline{\theta}_j(0)}{\partial v_a}$, $e_{ab}^{(j)} = \sqrt{T}\frac{\partial^2 \overline{\theta}_j(0)}{\partial v_a \partial v_b}$, and q is the dimension of the vector v.

Now the derivatives can be found by solving the following system of equations, for $j, k = 1, ..., 5$ and $a, b, c = 1, ..., q$:

$$\begin{aligned} 0 &= \sum_{k=1}^{5} A_{jk} e_a^{(k)} + \frac{\partial f_j\left(0,0\right)}{\partial v_a}, \\ 0 &= \sum_{k=1}^{5} \left(\frac{1}{\sqrt{T}} \sum_{l=1}^{5} K_{jkl} e_b^{(l)} + \frac{\partial^2 f_j\left(0,0\right)}{\partial v_b \partial \widetilde{\theta}_k} \right) e_a^{(k)} + \sum_{k=1}^{5} \frac{\partial^2 f_j\left(0,0\right)}{\partial v_a \partial \widetilde{\theta}_k} e_b^{(k)} \\ &\quad + \frac{1}{\sqrt{T}} \sum_{k=1}^{5} A_{jk} e_{ab}^{(k)}. \end{aligned}$$

The expected values of all the elements of the vector v, together with the derivatives e_a, e_{ab}, are then used in order to calculate appropriately the polynomial coefficients ψ_i in the Edgeworth expansion. Let us turn our attention to the theoretical bias approximations of the QMLEs.

3.4 Bias Approximations

In this section we develop the bias approximations for the QMLEs, under two different considerations. We distinguish between the case when the mean parameter is known, hence not estimated, and the case when this parameter is unknown and should be estimated with the other parameters of the model. One of the main advantages of developing the bias expressions is to use them as a bias correction mechanism. This is one of the practical applications of the bias approximations. A bias reduction is only possible in the case we are familiar with the exact bias expressions. Moreover, these results help to analyse the consequences of introducing restrictions in the log-variance parameters. With these expressions, one can compute the Edgeworth approximate distribution. It is important to explore the theoretical properties of the estimators so that the statistical inference is possible.

We use a McCullagh [20] result for the standardized estimator having a stochastic expansion, see in [20] p.209, and taking expectations we end up with the asymptotic bias of the QML estimator. Our next step is to check our bias approximations through simulation. Note that McCullagh's expansion has already been applied in the literature to retrieve the bias in many nonlinear models, such as Linton [17]. When dealing with nonlinear models, it is very common to have the bias expressions in terms of expectations and applying these expressions for bias correction. At this point, it is important to state briefly the main differences between our analysis and that of Linton. First of all, we generalize the finite-sample analysis of heteroskedastic time series models considering a non-symmetric distribution of the errors. Furthermore, we show that the block-diagonality of the information matrix does not hold in our case, which means that there are new terms in the bias expressions of the estimators.

Assumption 3.4.1 *We assume that the errors have third and fourth order cumulants, denoted by κ_3 and κ_4, respectively, where the latter is given by:*

$$\kappa_4 = E\left(z^4 - 3\right).$$

We further assume that the errors have bounded J^{th} moments, for some $J > 6$, so that the bias expressions exist.

Under the above assumptions, we are now able to present our Theorem which is useful for the evaluation of the bias approximations of all estimators and also to construct the Edgeworth expansions in this setting.

Theorem 3.4.1 *Given that $z_t \sim iid\,(0,1)$ and non-symmetric, and for $i, j, k \in \{\mu, \alpha, \theta, \gamma, \beta\}$ unless the parameter μ is used seperately to underline the difference, the following moments of the log-likelihood derivatives converge to finit limits as $T \to \infty$:*

$$c_{ij} = \tfrac{1}{T}E\left(\mathcal{L}_{ij}\right) = -\tfrac{1}{2}\tau_{i,j},$$

$$c_{ijk} = \tfrac{1}{T}E\left(\mathcal{L}_{ijk}\right) = -\tfrac{1}{2}\left(\tau_{ij,k} + \tau_{ik,j} + \tau_{jk,i} - \tau_{i,j,k}\right),$$

$$c_{ij,k} = \tfrac{1}{T}E\left(\mathcal{L}_{ij}\mathcal{L}_k\right) = -\tfrac{1}{4}\left[\tau^{zz}_{k;i,j} - \left(\kappa_4 + 2\right)\left(\tau_{ij,k} - \tau_{i,j,k}\right)\right],$$

$$c_{\mu\mu} = \tfrac{1}{T}E\left(\mathcal{L}_{\mu\mu}\right) = -\left(\overline{\pi} + \tfrac{\tau_{\mu,\mu}}{2}\right),$$

$$c_{i\mu\mu} = \tfrac{1}{T}E\left(\mathcal{L}_{i\mu\mu}\right) = \overline{\pi_i} - \tfrac{1}{2}\left(\tau_{i,\mu\mu} + 2\tau_{\mu i,\mu} - \tau_{\mu,i,\mu}\right),$$

$$c_{\mu\mu\mu} = \tfrac{1}{T}E\left(\mathcal{L}_{\mu\mu\mu}\right) = -\tfrac{1}{2}\left(3\tau_{\mu\mu,\mu} - \tau^3_{\mu}\right) + 3\overline{\pi_\mu},$$

$$c_{i\mu,\mu} = \tfrac{1}{T}E\left(\mathcal{L}_{i\mu}\mathcal{L}_\mu\right) = -\tfrac{1}{4}\left\{\begin{array}{c} 4\overline{\pi_i} - \left(\kappa_4 + 2\right)\left(\tau_{i\mu,\mu} - \tau_{i,\mu,\mu}\right) \\ +\tau^{zz}_{\mu;i\mu} + 2\tau^{zh}_{i,\mu} + 2\kappa_3\left(2\tau^h_{i,\mu} - \tau^h_{i\mu}\right) \end{array}\right\},$$

$$c_{i\mu,j} = \tfrac{1}{T}E\left(\mathcal{L}_{i\mu}\mathcal{L}_j\right) = -\tfrac{1}{4}\left\{-\left(\kappa_4 + 2\right)\left(\tau_{i\mu,j} - \tau_{i,j,\mu}\right) + \tau^{zz}_{\mu;i\mu} + 2\kappa_3\tau^h_{ij}\right\},$$

$c_{\mu\mu,i} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\mathcal{L}_i\right) = -\frac{1}{4}\left\{-\left(\kappa_4+2\right)\left(\tau_{\mu\mu,i}-\tau_{i,\mu,\mu}\right)+\tau^{zz}_{i;\mu\mu}+4\kappa_3\tau^h_{i,\mu}\right\}$,

$c_{ij,\mu} = \frac{1}{T}E\left(\mathcal{L}_{ij}\mathcal{L}_\mu\right) = -\frac{1}{4}\left\{\begin{array}{c} -\left(\kappa_4+2\right)\left(\tau_{ij,\mu}-\tau_{i,j,\mu}\right)+\tau^{zz}_{\mu;ij}+2\tau^{zh}_{i,j} \\ +2\kappa_3\left(2\tau^h_{i,j}-\tau^h_{ij}\right) \end{array}\right\}$,

$c_{\mu\mu,\mu} = \frac{1}{T}E\left(\mathcal{L}_{\mu\mu}\mathcal{L}_\mu\right) = -\frac{1}{4}\left\{\begin{array}{c} 8\overline{\pi_\mu}-\left(\kappa_4+2\right)\left(\tau_{\mu\mu,\mu}-\tau_{\mu,\mu,\mu}\right) \\ +\tau^{zz}_{\mu;\mu\mu}+2\tau^{zh}_{\mu,\mu}+2\kappa_3\left(3\tau^h_{\mu,\mu}-\tau^h_{\mu\mu}\right) \end{array}\right\}$,

where $\tau_i = \frac{1}{T}\sum_{t=1}^T E\left(h_{t;i}\right), \tau_{i,j} = \frac{1}{T}\sum_{t=1}^T E\left(h_{t;i}h_{t;j}\right), \tau_{ij,k} = \frac{1}{T}\sum_{t=1}^T E\left(h_{t;ij}h_{t;k}\right)$ *and* $\tau_{i,j,k} = \frac{1}{T}\sum_{t=1}^T E\left(h_{t;i}h_{t;j}h_{t;k}\right)$.

Also, $\overline{\pi} = \frac{1}{T}\sum_{t=1}^T E\left(\frac{1}{h_t}\right)$, *and* $\overline{\pi_i} = \frac{1}{T}\sum_{t=1}^T E\left(\frac{1}{h_t}h_{t;i}\right)$,

while $\tau^{zz}_{k;i,j} = \frac{1}{T}\sum\sum_{s<t} E\left[\left(z_s^2-1\right)h_{s;k}h_{t;i}h_{t;j}\right], \tau^{zh}_{i,j} = \frac{1}{T}\sum\sum_{s<t} E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;i}h_{t;j}\right)$,

$\tau^h_{i,\mu} = \frac{1}{T}\sum_{t=1}^T E\left(\frac{1}{\sqrt{h_t}}h_{t;i}h_{t;\mu}\right)$ *and* $\tau^h_{i\mu} = \frac{1}{T}\sum_{t=1}^T E\left(\frac{1}{\sqrt{h_t}}h_{t;i,\mu}\right)$.

Proof. Given in the Appendix. ■

The basic approach to finding bias approximations requires that we find expressions for the c^{ij}, c_{ijk} and $c_{jk,l}$. Let us first condider the case when the mean parameter is known and not estimated. With techniques of McCullagh [20], the standardized estimators, derived from choosing θ to solve $\mathcal{L}_i\left(\theta,\mu\right)=0$, for $i\in\left\{\alpha,\theta,\gamma,\beta\right\}$, have the following stochastic expansions^4:

$$\sqrt{T}\left\{\widehat{\theta}_i-\theta_i\right\} \approx -c^{ij}Z_j + \frac{1}{\sqrt{T}}\left\{c^{ij}c^{kl}Z_{jk}Z_l - c^{ij}c^{kl}c^{mn}c_{j\ln}Z_kZ_m/2\right\} + O_P\left(\frac{1}{T}\right), \tag{7}$$

where

$$Z_j = T^{-1/2}\mathcal{L}_j$$

and

$$Z_{jk} = T^{-1/2}\left\{\mathcal{L}_{jk}-E\left(\mathcal{L}_{jk}\right)\right\}$$

are evaluated at the true parameters and are jointly asymptotically normal. Raising pairs of indices signifies inversion, i.e. $c^{ij}=\left(c_{ij}\right)^{-1}$.

Taking expectations of the right-hand side in eq. (7), we get:

$$E\left[\sqrt{T}\varphi'\left\{\widehat{\theta}\left(\mu\right)-\theta\right\}\right] \approx \frac{1}{\sqrt{T}}\varphi_i c^{ij}c^{kl}\left\{c_{jk,l}+c_{jkl}\left(\kappa_4+2\right)/4\right\},$$

where φ is the 5×1 parameter vector. If $\kappa_4=0$, QML equals ML and then the above formula equals the one of Cox and Snell (1968), i.e.:

$$E\left[\sqrt{T}\varphi'\left\{\widehat{\theta}\left(\mu\right)-\theta\right\}\right]\approx\frac{1}{\sqrt{T}}\varphi_i c^{ij}c^{kl}\left\{c_{jk,l}+\frac{1}{2}c_{jkl}\right\}.$$

Let us now consider the other case, when the mean parameter is unknown and estimated with the other parameters. Hence, if we incorporate the effects of estimating μ, the stochastic expansions take the following form:

$$\sqrt{T}\left\{\widehat{\theta}_i\left(\widehat{\mu}\right)-\theta_i\right\}-\sqrt{T}\left\{\widehat{\theta}_i\left(\mu\right)-\theta_i\right\}\approx\frac{1}{\sqrt{T}}\left\{c^{ij}c^{kl}Z_{jk}Z_l-c^{ij}c^{kl}c^{mn}c_{j\ln}Z_kZ_m/2\right\},$$

where now $i,j,k,l\in\{\alpha,\theta,\gamma,\beta,\mu\}$. Taking expectations of the right-hand side, we find the asymptotic bias of the estimators in this case.

In terms of the mean squared error, from eq. (7) we have up to $O_P\left(\frac{1}{T}\right)$:

$$E\left[\sqrt{T}\varphi'\left\{\widehat{\theta}\left(\mu\right)-\theta\right\}\right]^2\approx-\varphi_i c^{ij}\left(\kappa_4+2\right)/2,\tag{8}$$

which is the asymptotic variance. If we let the remainder to be of $O\left(T^{-3/2}\right)$, then the mean squared error is again evaluated by eq. (8), with the difference now that there would be added terms of $O\left(T^{-1}\right)$. Of course, as $T\to\infty$, the mean squared error approaches the asymptotic variance.

4 Conclusions

Gaussian quasi-maximum likelihood estimation is a popular method which is widely used for inference in time series models. In this paper we study the asymptotic properties of the QMLEs in the EGARCH$(1,1)$ model of Nelson. The interest of establishing these theoretical properties has been growing but it remains a problem that awaits for an answer. In the current context, we present analytic derivatives both of the log-likelihood and the log-variance functions and also their expected values. We further develop theoretical bias approximations for the QMLEs of the parameters and we find conditions for stationarity of the log-variance derivatives. The theoretical results in this paper can be used to bias-correct the QMLEs in practice directly. In small or moderate-sized samples, a bias correction could be appreciable and it is helpful to have a rough estimate of its size.

The next steps in our research are to check the whole theoretical results through simulations and then examine the sensitivity of the bias approximations to different assumptions on the variance parameters. An interesting topic would be the investigation of necessary and sufficient conditions for the existence and validity of the Edgeworth approximations in this context. These issues are ongoing research.

Notes

[1] The validity of the Edgeworth expansions in the GARCH model is established in the paper of Corradi and Iglesias, [7].

[2] In a recent paper, Zaffaroni [31] estimates the EGARCH parameters with Whittle methods and the asymptotic distribution theory of these estimators is established.

[3] For an excellent review on the asymptotic expansion of the Edgeworth type, the reader is refered to Rothenberg, [24].

[4] We make use of the summation convention, that is: $c^{ij}Z_j = \sum_j c^{ij}Z_j$.

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Appendix A. Proof of the unconditional variance

We write the variance equation as follows:

$$\ln\left(h_t\right) = \alpha^* + \theta \sum_{i=0}^{\infty} \beta^i z_{t-1-i} + \gamma \sum_{i=0}^{\infty} \beta^i \left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right),$$

where $\alpha^* = \frac{\alpha}{1-\beta}$. Taking the expectation of the exponential of $\ln\left(h_t\right)$ we have:

$$E\exp\left(\ln h_t\right) = \exp\left(\alpha^*\right) E\exp\left[\textstyle\sum_{i=0}^{\infty}\left(\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right)\right] =$$

$$= \exp\left(\alpha^*\right) E\prod_{i=0}^{\infty}\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right] =$$

$$= \exp\left(\alpha^*\right)\prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right]$$

Now,

$$\prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left(\left|z_{t-1-i}\right| - E\left|z_{t-1-i}\right|\right)\right] =$$

$$= \prod_{i=0}^{\infty}\exp\left(-\gamma E\left|z_{t-1-i}\right|\beta^i\right) E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left|z_{t-1-i}\right|\right] =$$

$$= \exp\left(-\tfrac{\gamma E|z|}{1-\beta}\right)\prod_{i=0}^{\infty} E\exp\left[\theta\beta^i z_{t-1-i} + \gamma\beta^i\left|z_{t-1-i}\right|\right]$$

$$E\exp\left[\theta\beta^i z + \gamma\beta^i|z|\right] \overset{\substack{\kappa_1 = \theta\beta^i \\ \kappa_2 = \gamma\beta^i}}{=} E\exp\left[\kappa_1 z + \kappa_2|z|\right] = \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\kappa_1 z + \kappa_2|z| - \tfrac{1}{2}z^2\right)dz =$$

$$= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\tfrac{1}{2}\left(-2\left(\kappa_1-\kappa_2\right)z + z^2 \pm \left(\kappa_1-\kappa_2\right)^2\right)\right)dz$$

$$+\tfrac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\exp\left(-\tfrac{1}{2}\left(-2\left(\kappa_1+\kappa_2\right)z + z^2 \pm \left(\kappa_1+\kappa_2\right)^2\right)\right)dz =$$

$$= \tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\tfrac{1}{2}\left(-2\left(\kappa_1-\kappa_2\right)z + z^2 \pm \left(\kappa_1-\kappa_2\right)^2\right)\right)dz$$

$$+\tfrac{1}{\sqrt{2\pi}}\int_{0}^{\infty}\exp\left(-\tfrac{1}{2}\left(-2\left(\kappa_1+\kappa_2\right)z + z^2 \pm \left(\kappa_1+\kappa_2\right)^2\right)\right)dz =$$

$$= \exp\left(\tfrac{\left(\kappa_1-\kappa_2\right)^2}{2}\right)\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\tfrac{1}{2}\left(z-\left(\kappa_1-\kappa_2\right)\right)^2\right)dz$$

$$+\exp\left(\tfrac{\left(\kappa_1+\kappa_2\right)^2}{2}\right)\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left(-\tfrac{1}{2}\left(z-\left(\kappa_1+\kappa_2\right)\right)^2\right)dz =$$

$$= \exp\left(\tfrac{\left(\kappa_1-\kappa_2\right)^2}{2}\right)\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\left(\kappa_1-\kappa_2\right)}\exp\left(-\tfrac{1}{2}u^2\right)du + \exp\left(\tfrac{\left(\kappa_1+\kappa_2\right)^2}{2}\right)\tfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{-\left(\kappa_1+\kappa_2\right)}\exp\left(-\tfrac{1}{2}u^2\right)du =$$

$$= \exp\left(\tfrac{\left(\kappa_1-\kappa_2\right)^2}{2}\right)\Phi\left(-\left(\kappa_1-\kappa_2\right)\right) + \exp\left(\tfrac{\left(\kappa_1+\kappa_2\right)^2}{2}\right)\left(1-\Phi\left(-\left(\kappa_1+\kappa_2\right)\right)\right) =$$

$= \exp\left(\frac{(\kappa_1-\kappa_2)^2}{2}\right)\Phi\left(-\left(\kappa_1-\kappa_2\right)\right)+\exp\left(\frac{(\kappa_1+\kappa_2)^2}{2}\right)\Phi\left(\kappa_1+\kappa_2\right)=$

$= \exp\left(\frac{\beta^{2i}(\gamma-\theta)^2}{2}\right)\Phi\left(\beta^i\left(\gamma-\theta\right)\right)+\exp\left(\frac{\beta^{2i}(\gamma+\theta)^2}{2}\right)\Phi\left(\beta^i\left(\gamma+\theta\right)\right)$

$= \exp\left(\Delta\right)\Phi\left(-B\right)+\exp\left(\Gamma\right)\Phi\left(A\right),$

where $\Gamma=\frac{\beta^{2i}(\gamma+\theta)^2}{2}$, $\Delta=\frac{\beta^{2i}(\gamma-\theta)^2}{2}$, $A=\beta^i\left(\gamma+\theta\right)$ and $B=\beta^i\left(\gamma-\theta\right)$, and $\Phi\left(\cdot\right)$ is the cumulative distribution function of the standard normal distribution.

Therefore,

$$\begin{aligned}
E\exp\left(q\ln h_t\right) &= \exp\left(\frac{\alpha-\gamma E\left|z\right|}{1-\beta}\right)\prod_{i=0}^{\infty}\left(\begin{array}{c}\exp\left(\frac{\beta^{2i}(\gamma-\theta)^2}{2}\right)\Phi\left(\beta^i\left(\gamma-\theta\right)\right)\\ +\exp\left(\frac{\beta^{2i}(\gamma+\theta)^2}{2}\right)\Phi\left(\beta^i\left(\gamma+\theta\right)\right)\end{array}\right)\\
&= \exp\left(\Psi\right)\prod_{i=0}^{\infty}\left(\exp\left(\Delta\right)\Phi\left(-B\right)+\exp\left(\Gamma\right)\Phi\left(A\right)\right)\\
&= b^*,
\end{aligned}$$

where $\Psi=\frac{\alpha-\gamma E|z|}{1-\beta}$. ■

Appendix B. Expected values of the log-likelihood derivatives

The expected values of all first order derivatives are equal to zero.

Second order derivatives:

For $i,j\in\left\{\alpha,\theta,\gamma,\beta\right\}$,

$$\begin{aligned}
E\left(\mathcal{L}_{ij}\right) &= -\frac{T}{2}E\left(h_{t;i}h_{t;j}\right),\\
E\left(\mathcal{L}_{\mu j}\right) &= -\frac{T}{2}E\left(h_{t;\mu}h_{t;j}\right),\\
E\left(\mathcal{L}_{\mu\mu}\right) &= -TE\left(\frac{1}{h_t}\right)-\frac{T}{2}E\left(h_{t;\mu}^2\right).
\end{aligned}$$

Third order derivatives:

For $i\in\left\{\alpha,\theta,\gamma,\beta\right\}$,

$$E\left(\mathcal{L}_{iii}\right)=-\frac{T}{2}E\left(3h_{t;i}h_{t;i,i}-h_{t;i}^3\right),$$

for $i\in\left\{\alpha,\theta,\gamma,\beta\right\}, j\in\left\{\alpha,\theta,\gamma,\beta,\mu\right\}$,

$$E\left(\mathcal{L}_{iij}\right)=-\frac{T}{2}E\left(h_{t;j}h_{t;i,i}-h_{t;i}^2h_{t;j}+2h_{t;i}h_{t;i,j}\right),$$

for $i, j \in \{\alpha, \theta, \gamma, \beta\}, k \in \{\alpha, \theta, \gamma, \beta, \mu\}$,

$$E\left(\mathcal{L}_{ijk}\right) = -\frac{T}{2} E\left(h_{t;j} h_{t;i,k} + h_{t;k} h_{t;i,j} + h_{t;i} h_{t;j,k} - h_{t;j} h_{t;i} h_{t;k}, \right),$$

for $i \in \{\alpha, \theta, \gamma, \beta\}, j \in \{\mu\}$,

$$E\left(\mathcal{L}_{ijj}\right) = -\frac{T}{2} E\left(h_{t;i} h_{t;j,j} + 2h_{t;j} h_{t;i,j} - h_{t;i}\left(h_{t;j}\right)^2\right) + TE\left(\frac{1}{h_t} h_{t;i}\right),$$

for $j \in \{\mu\}$,

$$E\left(\mathcal{L}_{jjj}\right) = -\frac{T}{2} E\left(3h_{t;j} h_{t;j,j} - h_{t;j}^3\right) + TE\left(3\frac{1}{h_t} h_{t;j}\right).$$

In this Appendix we make a list of the results that are needed for the bias approximations. Please note that the last Appendix should be studied first in order to be familiarized with the symbols used.

First, provided that $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right| < 1$, the expected values of all second order derivatives are:

1. $E\left(\mathcal{L}_{\alpha\alpha}\right) = -\frac{T}{2}\frac{1+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\theta\gamma E(z|z|)\right)}$

2. $E\left(\mathcal{L}_{\alpha\beta}\right) = -\frac{T}{2}\frac{E(\ln(h_{t-1}))+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\theta\gamma E(z|z|)\right)}$

3. $E\left(\mathcal{L}_{\alpha\gamma}\right) = -\frac{T}{2}\frac{-\frac{1}{2}\left[E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

4. $E\left(\mathcal{L}_{\alpha\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

5. $E\left(\mathcal{L}_{\mu\alpha}\right) = -\frac{T}{2}\frac{-(\theta+\gamma EI)E\left(\frac{1}{\sqrt{h}}\right)+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\mu}+[\theta(\beta-\gamma E|z|)+\gamma\beta EI]E_{-\frac{1}{2}}E_{;\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

6. $E\left(\mathcal{L}_{\beta\beta}\right) = -\frac{T}{2}\frac{E\left(\ln^2(h_{t-1})\right)+2\left(\beta-\frac{1}{2}\gamma|z_{t-1}|\right)LE_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

7. $E\left(\mathcal{L}_{\beta\gamma}\right) = -\frac{T}{2}\frac{\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\gamma}-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

8. $E\left(\mathcal{L}_{\beta\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

9. $E\left(\mathcal{L}_{\beta\mu}\right) = -\frac{T}{2}\frac{-(\theta+\gamma EI)LE_{-\frac{1}{2}}+[\theta(\gamma E|z|-\beta)-\beta\gamma EI]E_{-\frac{1}{2}}E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\mu}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

10$E\left(\mathcal{L}_{\gamma\gamma}\right) = -\frac{T}{2}\frac{1-E^2|z|}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

11$E\left(\mathcal{L}_{\theta\gamma}\right) = -\frac{T}{2}\frac{E(z|z|)}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

12$E\left(\mathcal{L}_{\mu\gamma}\right) = -\frac{T}{2}\frac{-\gamma EI_{g(z)}E\left(\frac{1}{\sqrt{h}}\right)-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\mu}-(\theta(\beta-\gamma E|z|)+\gamma\beta EI)E_{-\frac{1}{2}}E_{;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

13$E\left(\mathcal{L}_{\theta\theta}\right) = -\frac{T}{2}\frac{1}{1-\left(\beta_2^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

14$E\left(\mathcal{L}_{\mu\theta}\right) = -\frac{T}{2}\frac{-\frac{1}{2}(\theta+\gamma E(z|z|))E(h_{t;\mu})-\gamma E|z|E\left(\frac{1}{\sqrt{h}}\right)+[\theta(\gamma E|z|-\beta)-\beta\gamma EI]E_{-\frac{1}{2}}E_{;\theta}}{1-\left(\beta_2^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$

15$E\left(\mathcal{L}_{\mu\mu}\right) = -TE\left(\frac{1}{h_t}\right) - \frac{T}{2}\frac{\left(\theta^2+\gamma^2+2\gamma\theta EI\right)E\left(\frac{1}{\sqrt{h_{t-1}}}\right)-2(\theta(\beta-\gamma E|z|)+\gamma\beta EI)E_{-\frac{1}{2}}E_{;\mu}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$. ■

Second, the expected values of the third order derivatives are:

1. $E\left(\mathcal{L}_{\alpha\alpha\alpha}\right) = -\frac{T}{2}E\left(3\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right) - h^3_{t;\alpha}\right)$

2. $E\left(\mathcal{L}_{\alpha\alpha\beta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\alpha} - h^2_{t;\alpha}h_{t;\beta} + 2h_{t;\alpha}h_{t;\alpha,\beta}\right)$

3. $E\left(\mathcal{L}_{\alpha\alpha\gamma}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\alpha,\alpha} - h^2_{t;\alpha}h_{t;\gamma} + 2h_{t;\alpha}h_{t;\alpha,\gamma}\right)$

4. $E\left(\mathcal{L}_{\alpha\alpha\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\alpha} - h^2_{t;\alpha}h_{t;\theta} + 2h_{t;\alpha}h_{t;\alpha,\theta}\right)$

5. $E\left(\mathcal{L}_{\mu\alpha\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha,\alpha}h_{t;\mu} + 2\left(h_{t;\alpha}h_{t;\mu,\alpha}\right) - h^2_{t;\alpha}h_{t;\mu}\right)$

6. $E\left(\mathcal{L}_{\beta\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\beta} + 2h_{t;\beta}h_{t;\beta,\alpha} - h_{t;\alpha}h^2_{t;\beta}\right)$

7. $E\left(\mathcal{L}_{\alpha\beta\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\beta} + h_{t;\alpha}h_{t;\beta,\gamma} - h_{t;\beta}h_{t;\alpha}h_{t;\gamma}\right)$

8. $E\left(\mathcal{L}_{\alpha\beta\theta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\alpha,\theta} + h_{t;\alpha}h_{t;\beta,\theta} + h_{t;\theta}h_{t;\alpha,\beta} - h_{t;\alpha}h_{t;\beta}h_{t;\theta}\right)$

9. $E\left(\mathcal{L}_{\mu\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\mu} + h_{t;\beta,\alpha}h_{t;\mu} + h_{t;\beta}h_{t;\mu,\alpha} - h_{t;\alpha}h_{t;\beta}h_{t;\mu}\right)$

10$E\left(\mathcal{L}_{\mu\beta\alpha}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\beta,\mu} + h_{t;\beta,\alpha}h_{t;\mu} + h_{t;\beta}h_{t;\mu,\alpha} - h_{t;\alpha}h_{t;\beta}h_{t;\mu}\right)$

11$E\left(\mathcal{L}_{\alpha\gamma\gamma}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\alpha,\gamma} - h_{t;\alpha}h^2_{t;\gamma}\right)$

12$E\left(\mathcal{L}_{\alpha\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\alpha,\gamma} + h_{t;\gamma}h_{t;\alpha,\theta} + h_{t;\alpha}h_{t;\gamma,\theta} - h_{t;\alpha}h_{t;\gamma}h_{t;\theta}\right)$

13$E\left(\mathcal{L}_{\alpha\gamma\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\gamma,\mu} - h_{t;\alpha}h_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\alpha}h_{t;\mu} + h_{t;\gamma}h_{t;\alpha,\mu}\right)$

14$E\left(\mathcal{L}_{\alpha\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\alpha,\theta} - h_{t;\alpha}h^2_{t;\theta}\right)$

15$E\left(\mathcal{L}_{\alpha\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\theta,\mu} - h_{t;\alpha}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\alpha}h_{t;\mu} + h_{t;\theta}h_{t;\alpha,\mu}\right)$

16$E\left(\mathcal{L}_{\alpha\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\alpha}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\alpha,\mu} - h_{t;\alpha}\left(h_{t;\mu}\right)^2\right) + TE\left(\frac{1}{h_t}h_{t;\alpha}\right)$

17$E\left(\mathcal{L}_{\beta\beta\beta}\right) = -\frac{T}{2}E\left(3h_{t;\beta}h_{t;\beta,\beta} - Eh^3_{t;\beta}\right)$

18$E\left(\mathcal{L}_{\beta\beta\gamma}\right) = -\frac{T}{2}E\left(2h_{t;\beta}h_{t;\gamma,\beta} + h_{t;\gamma}h_{t;\beta,\beta} - h^2_{t;\beta}h_{t;\gamma}\right)$

19$E\left(\mathcal{L}_{\beta\beta\theta}\right) = -\frac{T}{2}E\left(2h_{t;\beta}h_{t;\theta,\beta} + h_{t;\theta}h_{t;\beta,\beta} - h^2_{t;\beta}h_{t;\theta}\right)$

$20E\left(\mathcal{L}_{\beta\beta\mu}\right) = -\frac{T}{2}\left(2E\left(h_{t;\beta}h_{t;\beta,\mu}\right) + E\left(h_{t;\beta;\beta}h_{t;\mu}\right) - E\left(h_{t;\beta}^{2}h_{t;\mu}\right)\right)$

$21E\left(\mathcal{L}_{\beta\gamma\gamma}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\beta,\gamma} - h_{t;\beta}h_{t;\gamma}^{2}\right)$

$22E\left(\mathcal{L}_{\beta\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\beta,\gamma} + h_{t;\gamma}h_{t;\beta,\theta} + h_{t;\beta}h_{t;\gamma,\theta} - h_{t;\beta}h_{t;\gamma}h_{t;\theta}\right)$

$23E\left(\mathcal{L}_{\beta\gamma\mu}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\gamma,\mu} - h_{t;\beta}h_{t;\gamma}h_{t;\mu} + h_{t;\gamma,\beta}h_{t;\mu} + h_{t;\gamma}h_{t;\beta,\mu}\right)$

$24E\left(\mathcal{L}_{\beta\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\beta,\theta} - h_{t;\beta}h_{t;\theta}^{2}\right)$

$25E\left(\mathcal{L}_{\beta\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\theta,\mu} - h_{t;\beta}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\beta}h_{t;\mu} + h_{t;\theta}h_{t;\beta,\mu}\right)$

$26E\left(\mathcal{L}_{\mu\mu_{\beta}}\right) = -\frac{T}{2}E\left(h_{t;\beta}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\mu,\beta} - h_{t;\beta}\left(h_{t;\mu}\right)^{2}\right) + TE\left(\frac{1}{h_t}h_{t;\beta}\right)$

$27E\left(\mathcal{L}_{\gamma\gamma\gamma}\right) = -\frac{T}{2}E\left(3\left(h_{t;\gamma}h_{t;\gamma,\gamma}\right) - \left(h_{t;\gamma}^{3}\right)\right)$

$28E\left(\mathcal{L}_{\gamma\gamma\theta}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\gamma,\gamma} + 2h_{t;\gamma}h_{t;\gamma,\theta} - h_{t;\gamma}^{2}h_{t;\theta}\right)$

$29E\left(\mathcal{L}_{\gamma\gamma\mu}\right) = -\frac{T}{2}E\left(2h_{t;\gamma}h_{t;\gamma,\mu} - h_{t;\gamma}^{2}h_{t;\mu} + h_{t;\gamma,\gamma}h_{t;\mu}\right)$

$30E\left(\mathcal{L}_{\beta\theta\theta}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\theta} + 2h_{t;\theta}h_{t;\gamma,\theta} - h_{t;\gamma}h_{t;\theta}^{2}\right)$

$31E\left(\mathcal{L}_{\gamma\theta\mu}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\theta,\mu} - h_{t;\gamma}h_{t;\theta}h_{t;\mu} + h_{t;\theta,\gamma}h_{t;\mu} + h_{t;\theta}h_{t;\gamma,\mu}\right)$

$32E\left(\mathcal{L}_{\gamma\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\gamma}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\gamma,\mu} - h_{t;\gamma}h_{t;\mu}^{2}\right) + TE\left(\frac{1}{h_t}h_{t;\gamma}\right)$

$33E\left(\mathcal{L}_{\theta\theta\mu}\right) = -\frac{T}{2}E\left(2h_{t;\theta}h_{t;\theta,\mu} - h_{t;\theta}^{2}h_{t;\mu} + h_{t;\theta,\theta}h_{t;\mu}\right)$

$34E\left(\mathcal{L}_{\theta\mu\mu}\right) = -\frac{T}{2}E\left(h_{t;\theta}h_{t;\mu,\mu} + 2h_{t;\mu}h_{t;\theta,\mu} - h_{t;\theta}h_{t;\mu}^{2}\right) + TE\left(\frac{1}{h_t}h_{t;\theta}\right)$

$35E\left(\mathcal{L}_{\mu\mu\mu}\right) = -\frac{T}{2}E\left(3h_{t;\mu}h_{t;\mu,\mu} - h_{t;\mu}^{3}\right) + TE\left(3\frac{1}{h_t}h_{t;\mu}\right).\blacksquare$

Appendix C. Expected values of the log-variance derivatives

In the current Appendix, we present some of the results for the expected values of the log-variance derivatives and more specifically those that are needed for the evaluation of some of the expected values of the third order log-likelihood derivatives of the previous Appendix, that is:

Assuming first $\left|\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E\left|z\right| + \frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right| < 1$ and $\left|\beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 Ez^3 - \frac{3}{2}\gamma\left(\beta^2 E\left|z\right| - \beta\theta E\left(z\left|z\right|\right) + \frac{1}{4}\theta^2 E\left|z\right|^3\right) + \frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta Ez^3\right) - \frac{1}{8}\gamma^3 E\left|z\right|^3\right| < 1$, we have:

1. $E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\alpha)^2} + \left(\frac{1}{4}\beta\gamma E|z| - \frac{1}{8}\gamma^2 - \frac{1}{8}\theta^2 - \frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^3} + \left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\alpha,\alpha}}{1 - \left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)}$
2. $E\left(h_{t;\alpha}^{3}\right) = \frac{1 + 3\left(\beta - \frac{1}{2}\gamma E|z|\right)E_{;\alpha} + 3\left(\beta^2 + \frac{1}{4}\theta^2 + \frac{1}{4}\gamma^2 - \gamma\beta E|z| + \frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\alpha)^2}}{1 - \left[\beta^3 + \frac{3}{4}\beta\theta^2 - \frac{1}{8}\theta^3 Ez^3 - \frac{3}{2}\gamma\left(\beta^2 E|z| - \beta\theta E(z|z|) + \frac{1}{4}\theta^2 E|z|^3\right) + \frac{3}{4}\gamma^2\left(\beta - \frac{1}{2}\theta Ez^3\right) - \frac{1}{8}\gamma^3 E|z|^3\right]}$

$$3.\ E\left(h_{t;\beta}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\alpha)^2}+\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E_{(;\alpha)^2;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\alpha}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$4.\ E\left(h_{t;\alpha}h_{t;\alpha,\beta}\right)=\frac{E_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{(;\alpha)^2}+\frac{1}{4}\gamma E|z|E_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha,\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$5.\ E\left(h_{t;\gamma}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}[\theta E(z|z|)+\gamma(1-E|z|)]E_{(;\alpha)^2}-\frac{1}{2}[\theta E(z|z|)+\gamma(1-E|z|)]E_{;\alpha,\alpha}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\gamma}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$6.\ E\left(h_{t;\theta}h_{t;\alpha,\alpha}\right)=\frac{\frac{1}{4}[\theta+\gamma E(z|z|)]E_{(;\alpha)^2}-\frac{1}{2}[\theta+\gamma E(z|z|)]E_{;\alpha,\alpha}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$7.\ E\left(h_{t;\alpha}^2h_{t;\theta}\right)=\frac{-[\theta+\gamma E(z|z|)]E_{;\alpha}+\left(\frac{1}{2}\gamma\theta E|z|^3-\theta\beta+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E(z|z|)\right)E_{(;\alpha)^2}+2\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)E_{;\alpha;\theta}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$$

$$8.\ E\left(h_{t;\alpha}h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha,\theta}+\frac{1}{4}[\theta+\gamma E(z|z|)]E_{(;\alpha)^2}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{(;\alpha)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$9.\ E\left(h_{t;\alpha}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\beta)^2}+2E_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\beta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma|z_{t-1}|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{;\alpha(;\beta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$10E\left(h_{t;\beta}h_{t;\alpha,\beta}\right)=\frac{LE_{;\alpha}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\beta}+\frac{1}{4}\gamma E|z|LE_{;\alpha;\beta}+\left(\frac{1}{4}\beta\gamma|z_{t-1}|-\frac{1}{8}\gamma^2-\frac{1}{8}\theta^2-\frac{1}{4}\theta\gamma E(z|z|)\right)E_{;\alpha(;\beta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\beta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$11E\left(h_{t;\beta}h_{t;\alpha,\gamma}\right)=\frac{-\frac{1}{2}E|z|LE_{;\alpha}+\frac{1}{4}\gamma E|z|LE_{;\alpha;\gamma}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\gamma}-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{;\alpha}E_{;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\theta\gamma E(z|z|)\right)E_{;\alpha;\beta;}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$12E\left(h_{t;\beta}h_{t;\alpha,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{;\alpha;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\alpha,\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$13E\left(h_{t;\alpha}h_{t;\beta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{;\beta;\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta,\theta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$14E\left(h_{t;\theta}h_{t;\alpha,\beta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\beta}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha,\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\alpha;\theta}+\left(\frac{1}{4}\beta\gamma E|z|-\frac{1}{8}\theta^2-\frac{1}{8}\gamma^2-\frac{1}{4}\gamma\theta E(z|z|)\right)E_{;\alpha;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$15E\left(h_{t;\alpha}h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\gamma)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\gamma,\gamma}-\left(\beta E|z|-\frac{1}{2}\gamma-\frac{1}{2}\theta E(z|z|)\right)E_{;\alpha;\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\gamma)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$16E\left(h_{t;\alpha}h_{t;\gamma,\theta}\right)=\frac{-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{;\alpha;\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha;\gamma;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$17E\left(h_{t;\alpha}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|E_{(;\theta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\theta,\theta}+\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$18E\left(h_{t;\theta}h_{t;\alpha,\theta}\right)=\frac{-\frac{1}{2}E_{;\alpha}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\alpha,\theta}+\frac{1}{4}(\theta+\gamma E(z|z|))E_{;\alpha;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\alpha(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$19E\left(h_{t;\beta}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\beta)^2}+2LE_{;\beta}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\beta,\beta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{(;\beta)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\beta)^3}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$20E\left(h_{t;\beta}^3\right)=\frac{L^3+3\left(\beta-\frac{1}{2}\gamma E|z|\right)L^2E_{;\beta}+3\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)LE_{(;\beta)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$$

$$21E\left(h_{t;\theta}h_{t;\beta,\beta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\beta)^2}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\beta,\beta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\beta)^2;\theta}+2\left(\beta-\frac{1}{2}\gamma E|z|\right)E_{;\beta;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$22E\left(h_{t;\beta}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\gamma E|z|LE_{(;\theta)^2}+\left(\beta-\frac{1}{2}\gamma E|z|\right)LE_{;\theta,\theta}+\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\beta;\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\beta(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$23E\left(h_{t;\gamma}^3\right)=\frac{\left(E|z|^3-3E|z|+2E^3|z|\right)+3\left[\left(\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\right)\left(E|z|^3-E|z|\right)-\gamma\beta\left(1-E^2|z|\right)-\beta\theta E(z|z|)+\frac{1}{2}\gamma\theta\left(Ez^3-E|z|E(z|z|)\right)\right]E_{(;\gamma)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$$

$$24E\left(h_{t;\theta}h_{t;\gamma,\gamma}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\gamma)^2}-\frac{1}{2}(\theta+\gamma E(z|z|))E_{;\gamma,\gamma}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\gamma)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$25E\left(h_{t;\gamma}h_{t;\gamma,\theta}\right)=\frac{\frac{1}{4}(\theta+\gamma E(z|z|))E_{(;\gamma)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{(;\gamma)^2;\theta}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$26E\left(h_{t;\gamma}^2h_{t;\theta}\right)=\frac{Ez^3-2E|z|E(z|z|)+\left[\theta\left(\frac{1}{2}\gamma E|z|^3-\beta\right)+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E(z|z|)\right]E_{(;\gamma)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}$$

$$27E\left(h_{t;\gamma}h_{t;\theta,\theta}\right)=\frac{\frac{1}{4}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{(;\theta)^2}-\frac{1}{2}\left[\theta E(z|z|)+\gamma\left(1-E^2|z|\right)\right]E_{;\theta,\theta}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\gamma(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$28E\left(h_{t;\theta}h_{t;\gamma,\theta}\right)=\frac{-\frac{1}{2}\left(\beta E|z|-\frac{1}{2}\theta E(z|z|)-\frac{1}{2}\gamma\right)E_{(;\theta)^2}+\frac{1}{4}\left(\beta\gamma E|z|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\gamma\theta E(z|z|)\right)E_{;\gamma(;\theta)^2}}{1-\left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E|z|+\frac{1}{2}\gamma\theta E(z|z|)\right)}$$

$$29E\left(h_{t;\gamma}h_{t;\theta}^2\right)=\frac{E|z|^3-E|z|+\left[\left(\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2\right)\left(E|z|^3-E|z|\right)-\gamma\beta\left(1-E^2|z|\right)-\beta\theta E(z|z|)+\frac{1}{2}\gamma\theta\left(Ez^3-E|z|E(z|z|)\right)\right]E_{(;\theta)^2}}{1-\left[\beta^3+\frac{3}{4}\beta\theta^2-\frac{1}{8}\theta^3Ez^3-\frac{3}{2}\gamma\left(\beta^2E|z|-\beta\theta E(z|z|)+\frac{1}{4}\theta^2E|z|^3\right)+\frac{3}{4}\gamma^2\left(\beta-\frac{1}{2}\theta Ez^3\right)-\frac{1}{8}\gamma^3E|z|^3\right]}.\ \blacksquare$$

The whole results are available on demand from the corresponding author.

Appendix D. Expected values of cross products of the log-likelihood derivatives

In this Appendix, we present the expected values of cross-products of the log-likelihood derivatives. To conserve space, we present only some indicative. That is,

1. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\alpha}\right)=-\frac{1}{4}\left[\sum\sum_{s<t}E\left[\left(z_s^2-1\right)h_{s;\alpha}h_{t;\alpha}h_{t;\alpha}\right]-\left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\alpha}h_{t;\alpha,\alpha}-h_{t;\alpha}^3\right)\right]$
2. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\alpha\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum\sum_{s<t}E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}h_{t;\mu}\right]-\left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\alpha}h_{t;\alpha,\mu}-h_{t;\mu}h_{t;\alpha}^2\right)\\+2\kappa_3\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}^2\right)\end{array}\right]$
3. $\frac{1}{T}E\left(\mathcal{L}_\alpha\mathcal{L}_{\mu\mu}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum\sum_{s<t}E\left[\left(z_s^2-1\right)h_{s;\alpha}h_{t;\mu}^2\right]-\left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\alpha}h_{t;\mu,\mu}-h_{t;\alpha}h_{t;\mu}^2\right)\\+4\kappa_3\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}\right)\end{array}\right]$
4. $\frac{1}{T}E\left(\mathcal{L}_\mu\mathcal{L}_{\alpha\alpha}\right)=-\frac{1}{4}\left[\begin{array}{c}\sum\sum_{s<t}E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}^2\right]-\left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\mu}h_{t;\alpha,\alpha}-h_{t;\mu}h_{t;\alpha}^2\right)\\+2\sum\sum_{s<t}E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\alpha}^2\right)+2\kappa_3\sum_{t=1}^{T}E\left(\frac{1}{\sqrt{h_t}}h_{t;\alpha}^2-\frac{1}{\sqrt{h_t}}h_{t;\alpha,\alpha}\right)\end{array}\right]$

$$5.\ \tfrac{1}{T}E\left(\mathcal{L}_{\mu}\mathcal{L}_{\alpha\mu}\right)=-\tfrac{1}{4}\left[\begin{array}{c}\sum\sum\limits_{s<t}E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\alpha}h_{t;\mu}\right]-\left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\mu}h_{t;\alpha,\mu}-h_{t;\alpha}h_{t;\mu}^2\right)\\ +2\sum\sum\limits_{s<t}E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}\right)+2\kappa_3\sum_{t=1}^{T}E\left(\frac{2}{\sqrt{h_t}}h_{t;\alpha}h_{t;\mu}-\frac{1}{\sqrt{h_t}}h_{t;\alpha,\mu}\right)\\ +4\sum_{t=1}^{T}E\left(\frac{1}{h_t}h_{t;\alpha}\right)\end{array}\right]$$

$$6.\ \tfrac{1}{T}E\left(\mathcal{L}_{\mu}\mathcal{L}_{\mu\mu}\right)=-\tfrac{1}{4}\left[\begin{array}{c}\sum\sum\limits_{s<t}E\left[\left(z_s^2-1\right)h_{s;\mu}h_{t;\mu}^2\right]-\left(\kappa_4+2\right)\sum_{t=1}^{T}E\left(h_{t;\mu}h_{t;\mu,\mu}-h_{t;\mu}^3\right)\\ +2\sum\sum\limits_{s<t}E\left(z_s\frac{1}{\sqrt{h_t}}h_{t;\mu}^2\right)+2\kappa_3\sum_{t=1}^{T}E\left(\frac{3}{\sqrt{h_t}}h_{t;\mu}^2-\frac{1}{\sqrt{h_t}}h_{t;\mu,\mu}\right)\\ +8\sum_{t=1}^{T}E\left(\frac{1}{h_t}h_{t;\mu}\right)\end{array}\right]$$

At this point, we should note that these results differ from those in the paper of Linton [17], due to the fact that we assume non-symmetric distribution of the errors and also none of these expressions are zero, since the block-diagonality of the information matrix in our case that we study the EGARCH$(1,1)$ model does not hold.

Analytic proof of the first result is given as follows:

$$\begin{aligned}\mathcal{L}_{\alpha}\mathcal{L}_{\alpha\alpha} &= \frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\left(\frac{1}{2}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\alpha}-\frac{1}{2}\sum_{t=1}^{T}z_t^2\left(h_{t;\alpha}\right)^2\right)\\ &= \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha,\alpha}-\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{t=1}^{T}z_t^2h_{t;\alpha}^2\\ &= \frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)^2h_{t;\alpha}h_{t;\alpha,\alpha}+\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=t+1}^{T}\left(z_s^2-1\right)h_{s;\alpha,\alpha}\\ &\quad+\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=1}^{t-1}\left(z_s^2-1\right)h_{s;\alpha,\alpha}\\ &\quad-\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)z_t^2h_{t;\alpha}^3-\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=t+1}^{T}z_s^2h_{s;\alpha}^2-\frac{1}{4}\sum_{t=1}^{T}\left(z_t^2-1\right)h_{t;\alpha}\sum_{s=1}^{t-1}z_s^2h_{s;\alpha}^2\end{aligned}$$

Hence

$$E\left(\mathcal{L}_{\alpha}\mathcal{L}_{\alpha\alpha}\right)=\frac{T\left(\kappa_4+2\right)}{4}\left[E\left(h_{t;\alpha}h_{t;\alpha,\alpha}\right)-E\left(h_{t;\alpha}^3\right)\right]-\frac{1}{4}E\sum\sum_{s<t}\left(z_s^2-1\right)z_t^2h_{t;\alpha}^2h_{s;\alpha},$$

where

$h_{t;\alpha}=1+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}$ and $h_{t;\alpha}^2=1+2\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)h_{t-1;\alpha}+\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2h_{t-1;\alpha}^2$.

Let

$h_{t+k;\alpha} = 1 + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma \left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha}$ and $h^2_{t+k;\alpha} = 1 + 2\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma \left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma \left|z_{t+k-1}\right|\right)^2 h^2_{t+k-1;\alpha}$.

Hence,

$E\left[\left(z_t^2 - 1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] =$

$= E\left[\left(z_t^2 - 1\right)\left[h_{t;\alpha} + 2\left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma \left|z_{t+k-1}\right|\right) h_{t+k-1;\alpha} h_{t;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t+k-1} - \frac{1}{2}\gamma \left|z_{t+k-1}\right|\right)^2 h^2_{t+k-1;\alpha} h_{t;\alpha}\right]\right] =$

$\overset{k\geq 1}{=} 2\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right) E\left(z_t^2 - 1\right) h_{t+k-1;\alpha} h_{t;\alpha} + \left[\beta^2 + \frac{1}{4}\left(\theta^2 + \gamma^2\right) - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right] E\left(z_t^2 - 1\right) h^2_{t+k-1;\alpha} h_{t;\alpha}$.

$k = 1 : E\left[\left(z_t^2 - 1\right) h^2_{t+1;\alpha} h_{t;\alpha}\right] = E\left[\left(z_t^2 - 1\right)\left[h_{t;\alpha} + 2\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma \left|z_t\right|\right) h^2_{t;\alpha} + \left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma \left|z_t\right|\right)^2 h^3_{t;\alpha}\right]\right] =$

$= 2E\left[\left(z_t^2 - 1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma \left|z_t\right|\right)\right] Eh^2_{t;\alpha} + E\left[\left(z_t^2 - 1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma \left|z_t\right|\right)^2\right] Eh^3_{t;\alpha}$.

Hence,

$$E\left[\left(z_t^2 - 1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] \overset{k\geq 1}{=} 2\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\left[-\frac{1}{2}\theta E z^3 - \frac{1}{2}\gamma\left(E\left|z\right|^3 - E\left|z\right|\right)\right]\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-2} Eh^2_{t;\alpha}$$
$$+\left[\beta^2 + \frac{1}{4}\left(\theta^2 + \gamma^2\right) - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)\right] E\left(z_t^2 - 1\right) h^2_{t+k-1;\alpha} h_{t;\alpha}.$$

Set: $A = 2\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)\left[-\frac{1}{2}\theta E z^3 - \frac{1}{2}\gamma\left(E\left|z\right|^3 - E\left|z\right|\right)\right] Eh^2_{t;\alpha}$ and $C = \beta^2 + \frac{1}{4}\left(\theta^2 + \gamma^2\right) - \beta\gamma E\left|z\right| + \frac{1}{2}\theta\gamma E\left(z\left|z\right|\right)$.

We have that: $E\left[\left(z_t^2 - 1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] \overset{k\geq 1}{=} A\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-2} + CE\left(z_t^2 - 1\right) h^2_{t+k-1;\alpha} h_{t;\alpha}$.

By repeating substitution, $E\left[\left(z_t^2 - 1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] \overset{k\geq 1}{=} A\left[\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-2} + C\left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-3} + ... + C^{k-2}\right]$

$+C^{k-1} E\left(z_t^2 - 1\right) h^2_{t+1;\alpha} h_{t;\alpha}$.

This formula can be written as:

$E\left[\left(z_t^2 - 1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] \overset{k\geq 1}{=} A\frac{C^{k-1} - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-1}}{C - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)} + C^{k-1} E\left(z_t^2 - 1\right) h^2_{t+1;\alpha} h_{t;\alpha}$.

Consequently,

$E\left[\left(z_t^2 - 1\right) h^2_{t+k;\alpha} h_{t;\alpha}\right] \overset{k\geq 1}{=} A\frac{C^{k-1} - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)^{k-1}}{C - \left(\beta - \frac{1}{2}\gamma E\left|z\right|\right)}$

$+C^{k-1}\left[2E\left[\left(z_t^2 - 1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma \left|z_t\right|\right)\right] Eh^2_{t;\alpha} + E\left[\left(z_t^2 - 1\right)\left(\beta - \frac{1}{2}\theta z_t - \frac{1}{2}\gamma \left|z_t\right|\right)^2\right] Eh^3_{t;\alpha}\right]$. ■

Appendix E. Proof of the Theorem

The proof comes immediately from the results of Appendix B and Appendix D. ■

Appendix F. The log-variance derivatives

In this Appendix we present the expressions of the log-variance derivatives, in a form useful to explore their properties.

$$h_{t;\alpha} = 1 + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha}$$

$$h_{t;\alpha,\alpha} = \left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha}^{2} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha,\alpha}$$

$$h_{t;\alpha,\beta} = h_{t-1;\alpha} + \left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha} h_{t-1;\beta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha,\beta}$$

$$h_{t;\alpha,\gamma} = -\frac{1}{2}\left|z_{t-1}\right| h_{t-1;\alpha} + \left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha} h_{t-1;\gamma} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha,\gamma}$$

$$h_{t;\alpha,\theta} = -\frac{1}{2} z_{t-1} h_{t-1;\alpha} + \left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha} h_{t-1;\theta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha,\theta}$$

$$\begin{aligned} h_{t;\alpha,\mu} &= \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}} h_{t-1;\alpha} + \frac{1}{4}\left(\theta z_{t-1} + \gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha} h_{t-1;\mu} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\alpha,\mu} \end{aligned}$$

$$h_{t;\beta} = \ln\left(h_{t-1}\right) + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\beta}$$

$$h_{t;\beta,\beta} = \left(\frac{1}{4}\theta z_{t-1} + \frac{1}{4}\gamma \left|z_{t-1}\right|\right) h_{t-1;\beta}^{2} + 2h_{t-1;\beta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\beta,\beta}$$

$$h_{t;\beta,\gamma} = h_{t-1;\gamma} - \frac{1}{2}\left|z_{t-1}\right| h_{t-1;\beta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma \left|z_{t-1}\right|\right) h_{t-1;\beta} h_{t-1;\gamma} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\beta,\gamma}$$

$$h_{t;\beta,\theta} = h_{t-1;\theta} - \frac{1}{2} z_{t-1} h_{t-1;\beta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma \left|z_{t-1}\right|\right) h_{t-1;\beta} h_{t-1;\theta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\beta,\theta}$$

$$\begin{aligned} h_{t;\beta,\mu} &= h_{t-1;\mu} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}} h_{t-1;\beta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma \left|z_{t-1}\right|\right) h_{t-1;\beta} h_{t-1;\mu} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\beta,\mu} \end{aligned}$$

$$h_{t;\gamma} = g\left(z_{t-1}\right) + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma \left|z_{t-1}\right|\right) h_{t-1;\gamma}$$

$$h_{t;\gamma,\gamma} = -\left|z_{t-1}\right| h_{t-1;\gamma} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma}^{2} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma,\gamma}$$

$$h_{t;\gamma,\theta} = -\frac{1}{2}\left|z_{t-1}\right| h_{t-1;\theta} - \frac{1}{2} z_{t-1} h_{t-1;\gamma} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma} h_{t-1;\theta} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma,\theta}$$

$$\begin{aligned} h_{t;\gamma,\mu} &= -\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right] \frac{1}{\sqrt{h_{t-1}}} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}} h_{t-1;\gamma} \\ &\quad -\frac{1}{2}\left|z_{t-1}\right| h_{t-1;\mu} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma} h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\gamma,\mu} \end{aligned}$$

$$h_{t;\theta} = z_{t-1} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\theta}$$

$$h_{t;\theta,\theta} = -z_{t-1} h_{t-1;\theta} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\theta}^{2} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\theta,\theta}$$

$$\begin{aligned} h_{t;\theta,\mu} &= -\frac{1}{\sqrt{h_{t-1}}} + \frac{1}{2}\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}} h_{t-1;\theta} - \frac{1}{2} z_{t-1} h_{t-1;\mu} \\ &\quad + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\theta} h_{t-1;\mu} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\theta,\mu} \end{aligned}$$

$$h_{t;\mu} = -\left(\theta + \gamma\left[I\left(z_{t-1\geq 0}\right) - I\left(z_{t-1<0}\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\mu}$$

$$\begin{aligned} h_{t;\mu,\mu} &= \left(\theta + \gamma\left[I\left(z_{t-1} \geq 0\right) - I\left(z_{t-1} < 0\right)\right]\right) \frac{1}{\sqrt{h_{t-1}}} h_{t-1;\mu} + \frac{1}{4}\left(\theta z_{t-1} + \gamma\left|z_{t-1}\right|\right) h_{t-1;\mu}^{2} \\ &\quad + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\mu,\mu} \end{aligned}$$

Appendix G. Some additional calculations

How do we evaluate $E\left\{\exp\left[\kappa \ln h_t\right] h_{t;\alpha}^{2}\right\}$:

First, note that

$h_{t;\alpha}^{2} = 1 + 2\left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right) h_{t-1;\alpha} + \left(\beta - \frac{1}{2}\theta z_{t-1} - \frac{1}{2}\gamma\left|z_{t-1}\right|\right)^{2} h_{t-1;\alpha}^{2}$

$= \sum_{i=0}^{\infty}\left[1 + 2\left(\beta - \frac{1}{2}\theta z_{t-1-i} - \frac{1}{2}\gamma\left|z_{t-1-i}\right|\right) h_{t-1-i;\alpha}\right] \times \prod_{j=0}^{i-1}\left(\beta - \frac{1}{2}\theta z_{t-1-j} - \frac{1}{2}\gamma\left|z_{t-1-j}\right|\right)^{2}$

Hence,

$$E\left\{\exp\left[\kappa\ln h_t\right]h_{t;\alpha}^2\right\}=\textstyle\sum_{i=0}^{\infty}\displaystyle\prod_{j=0}^{i-1}E\left\{\left(\beta-\tfrac{1}{2}\theta z-\tfrac{1}{2}\gamma\left|z\right|\right)^2\exp\left[\kappa\beta^j\left(\alpha+\theta z+\gamma g\left(z\right)\right)\right]\right\}\times$$

$$\left\{E_{\kappa\beta^i}+2E\left\{\left(\beta-\tfrac{1}{2}\theta z-\tfrac{1}{2}\gamma\left|z\right|\right)\exp\left[\kappa\beta^i\left(\alpha+\theta z+\gamma g\left(z\right)\right)\right]\right\}E_{\kappa\beta^i}E_{;\alpha}\right\}$$

$$=E_\kappa E_{(;\alpha)^2}.\ \blacksquare$$

Additionally, we need:

$$\begin{aligned}
E\left(\exp\left(k\ln h_t\right)\ln\left(h_t\right)\right) &= E\left(\exp\left(k\ln h_t\right)\left(\frac{\alpha}{1-\beta}+\theta\sum_{s=0}^{\infty}\beta^s z_{t-1-s}+\gamma\sum_{s=0}^{\infty}\beta^s g\left(z_{t-1-s}\right)\right)\right)\\
&= \frac{\alpha}{1-\beta}E\left(\exp\left(k\ln h_t\right)\right)+E\left(\exp\left(k\ln h_t\right)\sum_{s=0}^{\infty}\beta^s\left(\theta z_{t-1-s}+\gamma g\left(z_{t-1-s}\right)\right)\right)\\
&= \frac{\alpha}{1-\beta}E\left(\exp\left(k\ln h_t\right)\right)\\
&\quad+\sum_{s=0}^{\infty}\beta^s E\left(\begin{array}{c}\exp\left(\frac{k\alpha}{1-\beta}+k\theta\sum_{j=0}^{\infty}\beta^j z_{t-1-j}+k\gamma\sum_{j=0}^{\infty}\beta^j g\left(z_{t-1-j}\right)\right)\\ \left(\theta z_{t-1-s}+\gamma g\left(z_{t-1-s}\right)\right)\end{array}\right)\\
&= \frac{\alpha}{1-\beta}E\left(\exp\left(k\ln h_t\right)\right)+\exp\left(\frac{k\alpha}{1-\beta}\right)\\
&\quad\times\sum_{s=0}^{\infty}\beta^s E\left(\exp\left(k\theta\beta^s z_{t-1-s}+k\gamma\beta^s g\left(z_{t-1-s}\right)\right)\left(\theta z_{t-1-s}+\gamma g\left(z_{t-1-s}\right)\right)\right)\\
&\quad\times\prod_{j=0\neq s}^{\infty}E\left(\exp\left(k\theta\beta^j z_{t-1-j}+k\gamma\beta^j g\left(z_{t-1-j}\right)\right)\right)\\
&= \frac{\alpha}{1-\beta}E\left(\exp\left(k\ln h_t\right)\right)+\exp\left(\frac{k\alpha}{1-\beta}\right)\prod_{j=0}^{\infty}E\left[\exp\left(k\theta\beta^j z+k\gamma\beta^j g\left(z\right)\right)\right]\\
&\quad\times\sum_{s=0}^{\infty}\beta^s\frac{E\left(\exp\left(k\theta\beta^s z+k\gamma\beta^s g\left(z\right)\right)\left[\theta z+\gamma g\left(z\right)\right]\right)}{E\left[\exp\left(k\theta\beta^s z+k\gamma\beta^s g\left(z\right)\right)\right]}\\
&= E_k L
\end{aligned}$$

Appendix H. Expected values of the first & second order log-variance derivatives

We assume $\left|\beta-\frac{1}{2}\gamma E\left|z\right|\right|<1$.

First order derivatives:

1. $E\left(h_{t;\alpha}\right) = \frac{1}{1-\beta+\frac{1}{2}\gamma E(|z|)}$
2. $E\left(h_{t;\beta}\right) = \frac{\alpha}{\left(1-\beta+\frac{\gamma}{2}E|z|\right)(1-\beta)}$
3. $E\left(h_{t;\gamma}\right) = 0$
4. $E\left(h_{t;\theta}\right) = 0$
5. $E\left(h_{t;\mu}\right) = -\frac{\theta E_{-\frac{1}{2}}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$. ■

Second order derivatives:

1. $E\left(h_{t;\alpha,\alpha}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\alpha)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
2. $E\left(h_{t;\alpha,\beta}\right) = \frac{E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
3. $E\left(h_{t;\alpha,\gamma}\right) = \frac{-\frac{1}{2}E|z|E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
4. $E\left(h_{t;\alpha,\theta}\right) = \frac{\frac{1}{4}\gamma E|z|E_{;\alpha;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
5. $E\left(h_{t;\alpha,\mu}\right) = \frac{\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\alpha}+\frac{1}{4}\gamma E|z|E_{;\alpha;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
6. $E\left(h_{t;\beta,\beta}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\beta)^2}+2E_{;\beta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
7. $E\left(h_{t;\beta,\gamma}\right) = \frac{-\frac{1}{2}E|z|E_{;\beta}+\frac{1}{4}\gamma E|z|E_{;\beta;\gamma}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
8. $E\left(h_{t;\beta,\theta}\right) = \frac{\frac{1}{4}\gamma E|z|E_{;\beta;\theta}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$
9. $E\left(h_{t;\beta,\mu}\right) = \frac{E_{;\mu}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\beta}+\frac{1}{4}\gamma E|z|E_{;\beta;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

10$E\left(h_{t;\gamma,\gamma}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\gamma)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

11$E\left(h_{t;\gamma,\theta}\right) = 0$

12$E\left(h_{t;\gamma,\mu}\right) = \frac{-EIE_{-\frac{1}{2}}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\gamma}-\frac{1}{2}E|z|E_{;\mu}+\frac{1}{4}\gamma E|z|E_{;\gamma;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

13$E\left(h_{t;\theta,\theta}\right) = \frac{\frac{1}{4}\gamma E|z|E_{(;\theta)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

14$E\left(h_{t;\theta,\mu}\right) = \frac{-E_{-\frac{1}{2}}+\frac{1}{2}(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\theta}+\frac{1}{4}\gamma E|z|E_{;\theta;\mu}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$

15$E\left(h_{t;\mu,\mu}\right) = \frac{(\theta+\gamma EI)E_{-\frac{1}{2}}E_{;\mu}+\frac{1}{4}\gamma E|z|E_{(;\mu)^2}}{1-\left(\beta-\frac{1}{2}\gamma E|z|\right)}$. ■

Appendix I. Useful formulae

Let I denote the Indicator function, then:

$$\begin{aligned}
E\left[\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\left(\theta z_{t-1}+\gamma g\left(z_{t-1}\right)\right)\right] &= \gamma\theta E\left|z\right|+\gamma^{2}EI_{g(z)} \\
E\left[\left(\theta+\gamma\left[I\left(z_{t-1}\geq 0\right)-I\left(z_{t-1}<0\right)\right]\right)\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\right] &= 2\theta\gamma E\left[\left|z\right|\right]
\end{aligned}$$

$$\begin{aligned}
E\left[\left(\theta z_{t-1}+\gamma g\left(z_{t-1}\right)\right)\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\right] &= \left[\theta^{2}+\gamma^{2}\left(1-E^{2}\left|z\right|\right)+2\theta\gamma E\left(z\left|z\right|\right)\right] \\
E\left[\left(\theta z_{t-1}+\gamma g\left(z_{t-1}\right)\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= -\frac{1}{2}\theta^{2}-\frac{1}{2}\gamma^{2}\left(1-E^{2}\left|z\right|\right)-\theta\gamma E\left(z\left|z\right|\right) \\
E\left[z_{t-1}\left(\theta z_{t-1}+\gamma g\left(z_{t-1}\right)\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= \left[\begin{array}{c} \beta\theta-\gamma\theta E\left|z\right|^{3}+\frac{1}{2}\gamma\theta E\left|z\right|-\frac{1}{2}\theta^{2}Ez^{3} \\ +\gamma\beta E\left(z\left|z\right|\right)-\frac{1}{2}\gamma^{2}\left(Ez^{3}-E\left|z\right|E\left(z\left|z\right|\right)\right) \end{array}\right] \\
E\left[\left(\theta z_{t-1}+\gamma g\left(z_{t-1}\right)\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^{2}\right] &= \left[\begin{array}{c} \frac{1}{4}\theta\left(\theta^{2}+3\gamma^{2}\right)E\left(z^{3}\right) \\ +\frac{1}{4}\gamma\left(\gamma^{2}+3\theta^{2}\right)E\left|z\right|^{3} \\ -\beta\theta^{2}-\beta\gamma^{2}-2\beta\gamma\theta E\left(z\left|z\right|\right) \\ -\frac{1}{4}\gamma\left(\theta^{2}+\gamma^{2}\right)E\left|z\right| \\ +\beta\gamma^{2}E^{2}\left|z\right|-\frac{1}{2}\gamma^{2}\theta E\left(z\left|z\right|\right)E\left|z\right| \end{array}\right]
\end{aligned}$$

$$\begin{aligned}
E\left[\left|z_{t-1}\right|^{2}g\left(z_{t-1}\right)\right] &= E\left[z_{t-1}^{2}g\left(z_{t-1}\right)\right]=E\left|z\right|^{3}-E\left|z\right| \\
E\left[g^{2}\left(z_{t-1}\right)\right] &= 1-E^{2}\left|z\right| \\
E\left[g^{3}\left(z_{t-1}\right)\right] &= E\left(\left|z_{t-1}\right|^{3}-3\left|z_{t-1}\right|^{2}E\left|z\right|+3\left|z_{t-1}\right|E^{2}\left|z\right|-E^{3}\left|z\right|\right) \\
&= \left(E\left|z\right|^{3}-3E\left|z\right|+2E^{3}\left|z\right|\right) \\
E\left[g^{2}\left(z_{t-1}\right)\left|z_{t-1}\right|\right] &= \left(E\left|z\right|^{3}-2E\left|z\right|+E^{3}\left|z\right|\right)
\end{aligned}$$

$$\begin{aligned}
E\left[\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= \theta\left(\beta-\gamma E\left|z\right|\right)+\gamma\beta EI \\
E\left[z_{t-1}\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= -\frac{1}{2}\theta^2-\frac{1}{2}\gamma^2 \\
&\quad -\gamma\theta E\left(z\left|z\right|\right)+\beta\gamma E\left|z\right| \\
E\left[g\left(z_{t-1}\right)\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= -\frac{1}{2}\theta^2 E\left(z\left|z\right|\right)+\gamma\theta E^2\left|z\right| \\
&\quad -\gamma\theta-\frac{1}{2}\gamma^2 EI_{z^2}-\gamma\beta E\left|z\right| EI \\
E\left[\left(\theta+\gamma\left[I\left(z_{t-1\geq 0}\right)-I\left(z_{t-1<0}\right)\right]\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2\right] &= \left[\begin{array}{c} \beta^2\theta+\frac{1}{4}\theta^3+\frac{3}{4}\gamma^2\theta \\ -2\beta\gamma\theta E\left|z\right|+\beta^2\gamma EI \\ +\frac{1}{4}\gamma\left(\gamma^2+3\theta^2\right)E\left(z\left|z\right|\right) \end{array}\right]
\end{aligned}$$

$$\begin{aligned}
E\left[g\left(z_{t-1}\right)z_{t-1}\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= \left[\begin{array}{c} -\frac{1}{2}\theta\left(E\left|z\right|^3-E\left|z\right|\right)+\beta E\left(z\left|z\right|\right) \\ -\frac{1}{2}\gamma\left(Ez^3-E\left|z\right|E\left(z\left|z\right|\right)\right) \end{array}\right] \\
E\left[g\left(z_{t-1}\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2\right] &= \left[\begin{array}{c} \frac{1}{4}\left(\theta^2+\gamma^2\right)\left(E\left|z\right|^3-E\left|z\right|\right)-\gamma\beta\left(1-E^2\left|z\right|\right) \\ -\beta\theta E\left(z\left|z\right|\right)+\frac{1}{2}\theta\gamma\left(Ez^3-E\left|z\right|E\left(z\left|z\right|\right)\right) \end{array}\right] \\
E\left[g^2\left(z_{t-1}\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= \left[\begin{array}{c} \beta\left(1-E^2\left|z\right|\right)-\frac{1}{2}\gamma\left(E\left|z\right|^3-2E\left|z\right|+E^3\left|z\right|\right) \\ -\frac{1}{2}\theta\left(Ez^3-2E\left|z\right|E\left(z\left|z\right|\right)\right) \end{array}\right] \\
E\left[z_{t-1}\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2\right] &= \theta\left(\frac{1}{2}\gamma E\left|z\right|^3-\beta\right)+\frac{1}{4}\left(\theta^2+\gamma^2\right)Ez^3-\beta\gamma E\left(z\left|z\right|\right) \\
E\left[\left(\theta z_{t-1}+\gamma\left|z_{t-1}\right|\right)\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)\right] &= \beta\gamma E\left|z\right|-\frac{1}{2}\gamma^2-\frac{1}{2}\theta^2-\theta\gamma E\left(z\left|z\right|\right)
\end{aligned}$$

$$\begin{aligned}
\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 &= \left(\begin{array}{c} \beta^2+\frac{1}{4}\theta^2 z_{t-1}^2+\frac{1}{4}\gamma^2\left|z_{t-1}\right|^2 \\ -\theta\beta z_{t-1}-\gamma\beta\left|z_{t-1}\right|+\frac{1}{2}\gamma\theta z_{t-1}\left|z_{t-1}\right| \end{array}\right) \\
E\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^2 &= \left(\beta^2+\frac{1}{4}\theta^2+\frac{1}{4}\gamma^2-\gamma\beta E\left|z\right|+\frac{1}{2}\gamma\theta E\left(z\left|z\right|\right)\right) \\
\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^3 &= \left(\begin{array}{c} \beta^3-\frac{3}{2}\beta^2\theta z_{t-1}+\frac{3}{4}\beta\theta^2 z_{t-1}^2-\frac{1}{8}\theta^3 z_{t-1}^3-\frac{3}{2}\beta^2\gamma\left|z_{t-1}\right|+\frac{3}{2}\beta\gamma\theta z_{t-1}\left|z_{t-1}\right| \\ -\frac{3}{8}\gamma\theta^2\left|z_{t-1}\right|^3+\frac{3}{4}\beta\gamma^2\left|z_{t-1}\right|^2-\frac{3}{8}\gamma^2\theta z_{t-1}^3-\frac{1}{8}\gamma^3\left|z_{t-1}\right|^3 \end{array}\right) \\
E\left(\beta-\frac{1}{2}\theta z_{t-1}-\frac{1}{2}\gamma\left|z_{t-1}\right|\right)^3 &= \left(\begin{array}{c} \beta^3+\frac{3}{4}\beta\theta^2+\frac{3}{4}\beta\gamma^2-\frac{1}{8}\theta\left(\theta^2+3\gamma^2\right)E\left(z^3\right)-\frac{3}{2}\beta^2\gamma E\left|z\right| \\ +\frac{3}{2}\beta\gamma\theta E\left(z\left|z\right|\right)-\frac{1}{8}\gamma\left(\gamma^2+3\theta^2\right)E\left|z\right|^3 \end{array}\right)
\end{aligned}$$

Appendix J. Symbols

The next symbols are used in the paper and more specifically in the expressions of the expected values of all the derivatives.

$$E\left(\exp\left(k\ln h_t\right)\ln\left(h_t\right)\right)=E_kL \quad E\left(\exp\left(k\ln h_t\right)\ln\left(h_{t-1}\right)\right)=E_kL(-1)$$

$$\begin{aligned}
E\left(\exp\left[\kappa\ln\left(h_t\right)\right]\exp\left[\lambda\ln\left(h_{t-1}\right)\right]\right) &= E_\kappa E_\lambda\left(-1\right)\\
E\Big(\exp\left[\kappa\ln\left(h_t\right)\right]\exp\left[\lambda\ln\left(h_{t-1}\right)\right]I_{z_{t-1}}\Big) &= E_\kappa EI_\lambda\left(-1\right)
\end{aligned}$$

$$\begin{aligned}
E\left(\exp\left(\kappa\ln h_t\right)\ln h_t\exp\left(\lambda\ln h_{t-1}\right)\right) &= E_\kappa LE_\lambda\left(-1\right)\\
E\Big(\exp\left[\kappa\ln\left(h_t\right)\right]\ln h_t\exp\left[\lambda\ln\left(h_{t-1}\right)\right]I_{z_{t-1}}\Big) &= E_\kappa LEI_\lambda\left(-1\right)\\
E\Big(\ln^2\left(h_t\right)\Big)=L^2 \quad E\left(\ln\left(h_t\right)\right)^3=L^3 \quad etc.
\end{aligned}$$

$$E\left(h_{t;\beta}\right)=E_{;\beta} \quad E\left(h_{t;\alpha}\right)=E_{;\alpha} \quad E\left(h_{t;\beta}\right)^2=E_{(;\beta)^2} \quad E\left(h_{t;\mu}\right)^3=E_{(;\mu)^3} \quad etc.$$

$$\begin{aligned}
E\left(\ln\left(h_t\right)h_{t;\mu}\right) &= LE_{;\mu} \quad E\left(\ln\left(h_t\right)h_{t;\alpha}\right)=LE_{;\alpha} \quad E\left(h_{t;\beta}\ln\left(h_t\right)\right)=LE_{;\beta} \quad etc\\
E\Big(h_{t;\beta}\left(\ln\left(h_t\right)\right)^2\Big) &= L^2E_{;\beta} \quad E\Big(\ln\left(h_t\right)h_{t;\beta}^2\Big)=LE_{(;\beta)^2}
\end{aligned}$$

$$\begin{aligned}
E\left(\exp\left(\kappa\ln h_t\right)h_{t;\beta}\right) &= E_kE_{;\beta} \quad E\left(\exp\left(k\ln h_t\right)h_{t;\alpha}\right)=E_kE_{;\alpha}\\
E\left(\exp\left(k\ln h_t\right)h_{t;\mu}\right) &= E_kE_{;\mu} \quad etc.\\
E\Big(\exp\left(\kappa\ln h_t\right)h_{t;\mu}^2\Big) &= E_\kappa E_{(;\mu)^2} \quad E\Big(\exp\left(\kappa\ln h_t\right)h_{t;\mu}^3\Big)=E_\kappa E_{(;\mu)^3} \quad etc.\\
E\Big(h_{t;\beta}\left(h_{t;\mu}\right)^2\Big) &= E_{;\beta(;\mu)^2}
\end{aligned}$$

$$\begin{aligned}
E\left(h_{t-1;\beta}h_{t-1;\mu}\right) &= E_{;\beta;\mu} \quad E\left(h_{t;\beta}h_{t;\alpha}\right)=E_{;\beta;\alpha} \quad E\left(h_{t;\beta}h_{t;\gamma}\right)=E_{;\beta;\gamma} \quad etc.\\
E\left(\exp\left[\kappa\ln\left(h_t\right)\right]h_{t;\beta}h_{t;\mu}\right) &= E_\kappa E_{;\beta;\mu} \quad etc\\
E\left(h_{t;\beta,\mu}\right)=E_{;\beta,\mu} \quad E\left(h_{t;\mu,\mu}\right)=E_{;\mu,\mu} \quad etc.
\end{aligned}$$

$$\begin{aligned}
E\left(h_{t;\mu}h_{t;\mu,\mu}\right) &= E_{;\mu;\mu,\mu}\\
E\left(\ln\left(h_t\right)h_{t;\beta,\beta}\right) &= LE_{;\beta,\beta}\\
E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu,\mu}\right) &= E_\kappa E_{;\mu,\mu}\\
E\left(h_{t;\beta}h_{t;\beta,\beta}\right) &= E_{;\beta;\beta,\beta}\\
E\left(\exp\left(\kappa\ln h_t\right)\ln\left(h_t\right)h_{t;\mu}\right) &= E_\kappa LE_{;\mu}
\end{aligned}$$

$$\begin{aligned}
E\left(h_{t;\beta}h_{t;\mu,\mu}\right) &= E_{;\beta;\mu,\mu} \quad E\left(h_{t;\mu}h_{t;\mu,\beta}\right)=E_{;\mu;\mu,\beta}\\
E\left(\exp\left(\kappa\ln h_t\right)h_{t;\mu,\beta}\right) &= E_\kappa E_{;\mu,\beta}
\end{aligned}$$